Separable Differential Equations

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Definition (Separable Equation). An equation $y' = f(x, y)$ is called **separable** provided there exists functions $F(x)$ and $G(y)$ such that

$$f(x, y) = F(x)G(y).$$

Definition (Separated Form of a Separable Equation). The equation

$$\frac{y'}{G(y)} = F(x).$$

is called the **separated form**. It is obtained from the separable equation $y' = F(x)G(y)$ by dividing by $G(y)$.

Such an equation is said to be *prepared for quadrature*, because the left side is independent of x and the right side is independent of y, y'.

Finding a Separable Form

The algorithm supplied here determines F and G such that $f(x, y) = F(x)G(y)$. The algorithm also applies to **prove** that an equation is **not separable**.

Algorithm. Given differential equation $y' = f(x, y)$, **invent values** x_0, y_0 such that $f(x_0, y_0) \neq 0$. Define F, G by the formulas

$$F(x) = \frac{f(x, y_0)}{f(x_0, y_0)}, \quad G(y) = f(x_0, y). \tag{1}$$

Because $f(x_0, y_0) \neq 0$, then (1) makes sense. Test I *infra* implies the following test.

Theorem 1 (Separability Test)
Let F and G be defined by (1). Multiply FG. Then
(a) If $F(x)G(y) = f(x, y)$, then $y' = f(x, y)$ is **separable**.
(b) If $F(x)G(y) \neq f(x, y)$, then $y' = f(x, y)$ is **not separable**.

Example 1: Let $f(x,y) = 6xy + 8y - 15x - 20$. Find F and G in Relation $f(x,y) = F(x)G(y)$. ________________________________

Answer: $F = 3x + 4$, $G = 2y - 5$.
Start with $f(x,y) = F(x)G(y)$ and substitute $y = 0$ (use another value for y, like $y = 1$, if this fails). Then $f(x,0) = (6xy + 8y - 15x - 20)\big|_{y=0} = -15x - 20$ which implies $F(x)G(0) = -15x - 20$. Assume $G(0)$ is the constant 1, then $F(x) = -15x - 20$.

Repeat with $x = 0$ in the relation $F(x)G(y) = 6xy + 8y - 15x - 20$ to get $F(0)G(y) = 8y - 20$. Because $F(x) = -15x - 20$ then $F(0) = -20$. The previous relation becomes $(-20)G(y) = 8y - 20$ or $G(y) = -\frac{2}{5}y + 1$.

We don't know by this analysis if F and G actually work, that is, if $F(x)G(y)$ multiplies out to give $f(x,y) = 6xy + 8y - 15x - 20$. So we check it:

$$\begin{aligned}
F(x)G(y) &= (-15x - 20)\left(-\tfrac{2}{5}y + 1\right) \\
&= (-3x - 4)(-2y + 5) \quad \text{(cancel common 5)} \\
&= (3x + 4)(2y - 5) \quad \text{(adjust minus sign)} \\
&= 6xy + 8y - 15x - 20 \\
&= f(x,y).
\end{aligned}$$

Therefore it works. We can adjust F and G by constants that cancel, so we choose $F = 3x + 4$ and $G = 2y - 5$, as discovered in the check above.

Example 2: Assume $f(x,y) = 6xy + 8y - 15x - 20$. Find F and G in Relation $f(x,y) = F(x)G(y)$

We determine without factorization talent the formula $f(x,y) = (3x+4)(2y-5)$. Invent values $x_0 = 0$, $y_0 = 0$, chosen to make $f(x_0, y_0) = -20$ nonzero. Define

$$\begin{aligned} F(x) &= \frac{f(x,y_0)}{f(x_0,y_0)} = \frac{0+0-15x-20}{-20} = \frac{3}{4}x + 1 \\ G(y) &= f(x_0,y) = 0 + 8y - 0 - 20 = 8y - 20. \end{aligned}$$

Then $f(x,y) = F(x)G(y)$ because

$$F(x)G(y) = \left(\frac{3}{4}x + 1\right)(8y - 20) = 6xy + 8y - 15x - 20.$$

Because $cF(x)(1/c)G(y) = f(x,y)$ for any $c \neq 0$, we can choose $c = 4$ to get a simpler fraction-free factorization using the new definitions $F = 3x+4$, $G = 2y-5$. The final answers:

$$F = 3x + 4, \quad G = 2y - 5$$

Non-Separability Tests

Test I. Equation $y' = f(x, y)$ is not separable provided for some pair of points (x_0, y_0), (x, y) in the domain of f, (2) holds:

$$f(x, y_0)f(x_0, y) - f(x_0, y_0)f(x, y) \neq 0. \tag{2}$$

Test II. The equation $y' = f(x, y)$ is not separable if either of the following conditions hold:

- $f_x(x, y)/f(x, y)$ is non-constant in y

  or

- $f_y(x, y)/f(x, y)$ is non-constant in x.

Test I details

Assume $\boldsymbol{f(x,y) = F(x)G(y)}$, then equation (2) fails because each term on the left side of (2) equals $\boldsymbol{F(x)G(y_0)F(x_0)G(y)}$ for all choices of $\boldsymbol{(x_0, y_0)}$ and $\boldsymbol{(x, y)}$ (hence contradiction $\boldsymbol{0 \neq 0}$).

Test II details

Assume $\boldsymbol{f(x,y) = F(x)G(y)}$ and suppose $\boldsymbol{F}$, $\boldsymbol{G}$ are sufficiently differentiable. Then

- $\boldsymbol{\dfrac{f_x(x,y)}{f(x,y)} = \dfrac{F'(x)}{F(x)}}$ is independent of $\boldsymbol{y}$

  and
- $\boldsymbol{\dfrac{f_y(x,y)}{f(x,y)} = \dfrac{G'(y)}{G(y)}}$ is independent of $\boldsymbol{x}$.

Illustration

Consider $y' = xy + y^2$.

Test I implies it is not separable, because the left side of the relation is

$$\begin{aligned} \text{LHS} &= f(x,1)f(0,y) - f(0,1)f(x,y) \\ &= (x+1)y^2 - (xy + y^2) \\ &= x(y^2 - y) \\ &\neq 0. \end{aligned}$$

Test II implies it is not separable, because

$$\frac{f_x}{f} = \frac{1}{x+y}$$

is not constant as a function of y.

Variables-Separable Method

The method determines two kinds of solution formulas.

Equilibrium Solutions.

They are the constant solutions $y = c$ of $y' = f(x, y)$. For any equation, find them by substituting $y = c$ into the differential equation.

Non-Equilibrium Solutions.

For a separable equation

$$y' = F(x)G(y),$$

a non-equilibrium solution y is a solution with $G(y) \neq 0$. It is found by dividing by $G(y)$, then applying the method of quadrature.

Theory of Non-Equilibrium Solutions

A given solution $y(x)$ satisfying $G(y(x)) \neq 0$ throughout its domain of definition is called a non-equilibrium solution. Then division by $G(y(x))$ is allowed.

The *method of quadrature* applies to the separated equation $y'/G(y(x)) = F(x)$. Some details:

$\int_{x_0}^{x} \frac{y'(t)dt}{G(y(t))} = \int_{x_0}^{x} F(t)dt$	Integrate both sides of the separated equation over $x_0 \le t \le x$.
$\int_{y_0}^{y(x)} \frac{du}{G(u)} = \int_{x_0}^{x} F(t)dt$	Apply on the left the change of variables $u = y(t)$. Define $y_0 = y(x_0)$.
$y(x) = M^{-1}\left(\int_{x_0}^{x} F(t)dt\right)$	Define $M(y) = \int_{y_0}^{y} du/G(u)$. Take inverses to isolate $y(x)$.

In practise, the last step with M^{-1} is never done. The preceding formula is called the *implicit solution*. Some work is done to find algebraically an *explicit solution*, as is given by W^{-1}.

Explicit and Implicit Solutions

Definition 1 (Explicit Solution)
A solution y of $y' = f(x, y)$ is called **explicit** provided it is given by an equation

$$y = \text{an expression independent of } y.$$

To elaborate, on the left side must appear exactly the symbol y, followed by an equal sign. Symbols y and $=$ are followed by an expression which does not contain the symbol y.

Definition 2 (Implicit Solution)
A solution of $y' = f(x, y)$ is called **implicit** provided it is not explicit.

Examples

- Explicit solutions: $y = 1$, $y = x$, $y = f(x)$, $y = 0$, $y = -1 + x^2$
- Implicit Solutions: $2y = 2$, $y^2 = x$, $y + x = 0$, $y = xy^2 + 1$, $y + 1 = x^2$, $x^2 + y^2 = 1$, $F(x, y) = c$

The General Solution of $y' = 2x(y-3)$

- The variables-separable method gives equilibrium solutions $y = c$, which are already *explicit*. In this case, $y = 3$ is an equilibrium solution.
- Because $F = 2x$, $G = y - 3$, then division by G gives the quadrature-prepared equation $y'/(y-3) = 2x$. A quadrature step gives the implicit solution

$$\ln|y-3| = x^2 + C.$$

- The non-equilibrium solutions may be left in *implicit* form, giving the **general solution** as the list

$$L_1 = \{y = 3, \ln|y-3| = x^2 + C\}.$$

- Algebra can be applied to $\ln|y-3| = x^2 + C$ to write it as $y = 3 + ke^{x^2}$ where $k \neq 0$. Then general solution L_1 can be re-written as

$$L_2 = \{y = 3, y = 3 + ke^{x^2}\}.$$

  List L_2 can be distilled to the single formula $y = 3 + ce^{x^2}$, but L_1 has no simpler expression.

Answer Check an Explicit Solution

To answer check $y' = 1 + y$ with **explicit solution** $y = -1 + ce^x$, expand the left side of the DE and the right side of the DE separately, then compare the two computations.

$\text{LHS} = y'$	Left side of the DE.
$= (-1 + ce^x)'$	Substitute the solution $y = -1 + ce^x$.
$= 0 + ce^x$	Evaluate.
$\text{RHS} = 1 + y$	Right side of the DE.
$= -1 + 1 + ce^x$	Substitute the solution $y = -1 + ce^x$.
$= ce^x$	Evaluate.

Then LHS $=$ RHS for all symbols. The DE is verified.

Answer Check an Implicit Solution

To answer check

$$y' = 1 + y^2$$

with **implicit solution**

$$\arctan(y) = x + c,$$

differentiate the implicit solution equation on x, to produce the differential equation.

$\arctan(y(x)) = x + c$	The implicit equation, replacing y by $y(x)$.
$\frac{d}{dx}\arctan(y(x)) = \frac{d}{dx}(x + c)$	Differentiate the previous equation.
$\frac{y'(x)}{1 + (y(x))^2} = 1 + 0$	Chain rule applied left.
$y' = (1 + 0)(1 + y^2)$	Cross-multiply to isolate y' left.
$y' = 1 + y^2$	The DE is verified.