11.8 Second-order Systems

A model problem for second order systems is the system of three masses coupled by springs studied in section 11.1, equation (6):

$$\begin{aligned} m_1 x_1''(t) &= -k_1 x_1(t) + k_2[x_2(t) - x_1(t)], \\ m_2 x_2''(t) &= -k_2[x_2(t) - x_1(t)] + k_3[x_3(t) - x_2(t)], \\ m_3 x_3''(t) &= -k_3[x_3(t) - x_2(t)] - k_4 x_3(t). \end{aligned} \tag{1}$$

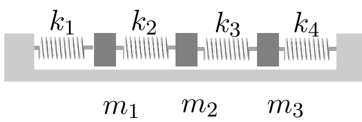

Figure 20. Three masses connected by springs. The masses slide along a frictionless horizontal surface.

In vector-matrix form, this system is a **second order system**

$$M\mathbf{x}''(t) = K\mathbf{x}(t)$$

where the **displacement x**, **mass matrix** M and **stiffness matrix** K are defined by the formulas

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}.$$

Because M is invertible, the system can always be written as

$$\mathbf{x}'' = A\mathbf{x}, \quad A = M^{-1}K.$$

Converting $\mathbf{x}'' = A\mathbf{x}$ to $\mathbf{u}' = C\mathbf{u}$

Given a second order $n \times n$ system $\mathbf{x}'' = A\mathbf{x}$, define the variable $\mathbf{u}$ and the $2n \times 2n$ block matrix C as follows.

$$\mathbf{u} = \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix}, \quad C = \left(\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right). \tag{2}$$

Then each solution $\mathbf{x}$ of the second order system $\mathbf{x}'' = A\mathbf{x}$ produces a corresponding solution $\mathbf{u}$ of the first order system $\mathbf{u}' = C\mathbf{u}$. Similarly, each solution $\mathbf{u}$ of $\mathbf{u}' = C\mathbf{u}$ gives a solution $\mathbf{x}$ of $\mathbf{x}'' = A\mathbf{x}$ by the formula $\mathbf{x} = \textbf{diag}(I, 0)\mathbf{u}$.

Characteristic Equation for $\mathbf{x}'' = A\mathbf{x}$

The characteristic equation for the $n \times n$ second order system $\mathbf{x}'' = A\mathbf{x}$ can be obtained from the corresponding $2n \times 2n$ first order system $\mathbf{u}' = C\mathbf{u}$. We will prove the following identity.

Theorem 31 (Characteristic Equation)
Let $\mathbf{x}'' = A\mathbf{x}$ be given with A $n \times n$ constant and let $\mathbf{u}' = C\mathbf{u}$ be its corresponding first order system, using (2). Then

$$\det(C - \lambda I) = (-1)^n \det(A - \lambda^2 I). \tag{3}$$

Proof: The method of proof is to verify the product formula

$$\left(\begin{array}{c|c} -\lambda I & I \\ \hline A & -\lambda I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline \lambda I & I \end{array}\right) = \left(\begin{array}{c|c} 0 & I \\ \hline A - \lambda^2 I & -\lambda I \end{array}\right).$$

Then the determinant product formula applies to give

$$\det(C - \lambda I)\det\left(\begin{array}{c|c} I & 0 \\ \hline \lambda I & I \end{array}\right) = \det\left(\begin{array}{c|c} 0 & I \\ \hline A - \lambda^2 I & -\lambda I \end{array}\right). \tag{4}$$

Cofactor expansion is applied to give the two identities

$$\det\left(\begin{array}{c|c} I & 0 \\ \hline \lambda I & I \end{array}\right) = 1, \quad \det\left(\begin{array}{c|c} 0 & I \\ \hline A - \lambda^2 I & -\lambda I \end{array}\right) = (-1)^n \det(A - \lambda^2 I).$$

Then (4) implies (3). The proof is complete.

Solving $\mathbf{u}' = C\mathbf{u}$ and $\mathbf{x}'' = A\mathbf{x}$

Consider the $n \times n$ second order system $\mathbf{x}'' = A\mathbf{x}$ and its corresponding $2n \times 2n$ first order system

$$\mathbf{u}' = C\mathbf{u}, \quad C = \left(\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right), \quad \mathbf{u} = \left(\begin{array}{c} \mathbf{x} \\ \mathbf{x}' \end{array}\right). \tag{5}$$

Theorem 32 (Eigenanalysis of A and C)
Let A be a given $n \times n$ constant matrix and define the $2n \times 2n$ block matrix C by (5). Then

$$(C - \lambda I)\left(\begin{array}{c} \mathbf{w} \\ \mathbf{z} \end{array}\right) = \mathbf{0} \quad \text{if and only if} \quad \begin{cases} A\mathbf{w} = \lambda^2 \mathbf{w}, \\ \mathbf{z} = \lambda \mathbf{w}. \end{cases} \tag{6}$$

Proof: The result is obtained by block multiplication, because

$$C - \lambda I = \left(\begin{array}{c|c} -\lambda I & I \\ \hline A & -\lambda I \end{array}\right).$$

Theorem 33 (General Solutions of $\mathbf{u}' = C\mathbf{u}$ and $\mathbf{x}'' = A\mathbf{x}$)
Let A be a given $n \times n$ constant matrix and define the $2n \times 2n$ block matrix C by (5). Assume C has eigenpairs $\{(\lambda_j, \mathbf{y}_j)\}_{j=1}^{2n}$ and $\mathbf{y}_1, \ldots, \mathbf{y}_{2n}$ are independent. Let I denote the $n \times n$ identity and define $\mathbf{w}_j = \textbf{diag}(I, 0)\mathbf{y}_j$, $j = 1, \ldots, 2n$. Then $\mathbf{u}' = C\mathbf{u}$ and $\mathbf{x}'' = A\mathbf{x}$ have general solutions

$$\begin{aligned} \mathbf{u}(t) &= c_1 e^{\lambda_1 t}\mathbf{y}_1 + \cdots + c_{2n} e^{\lambda_{2n} t}\mathbf{y}_{2n} \qquad (2n \times 1), \\ \mathbf{x}(t) &= c_1 e^{\lambda_1 t}\mathbf{w}_1 + \cdots + c_{2n} e^{\lambda_{2n} t}\mathbf{w}_{2n} \qquad (n \times 1). \end{aligned}$$

Proof: Let $\mathbf{x}_j(t) = e^{\lambda_j t}\mathbf{w}_j$, $j = 1, \ldots, 2n$. Then $\mathbf{x}_j$ is a solution of $\mathbf{x}'' = A\mathbf{x}$, because $\mathbf{x}_j''(t) = e^{\lambda_j t}(\lambda_j)^2\mathbf{w}_j = A\mathbf{x}_j(t)$, by Theorem 32. To be verified is the independence of the solutions $\{\mathbf{x}_j\}_{j=1}^{2n}$. Let $\mathbf{z}_j = \lambda_j \mathbf{w}_j$ and apply Theorem 32 to write $\mathbf{y}_j = \begin{pmatrix} \mathbf{w}_j \\ \mathbf{z}_j \end{pmatrix}$, $A\mathbf{w}_j = \lambda_j^2 \mathbf{w}_j$. Suppose constants $a_1, \ldots, a_{2n}$ are given such that $\sum_{j=1}^{2n} a_k \mathbf{x}_j = 0$. Differentiate this relation to give $\sum_{j=1}^{2n} a_k e^{\lambda_j t}\mathbf{z}_j = 0$ for all t. Set $t = 0$ in the last summation and combine to obtain $\sum_{j=1}^{2n} a_k \mathbf{y}_j = 0$. Independence of $\mathbf{y}_1, \ldots, \mathbf{y}_{2n}$ implies that $a_1 = \cdots = a_{2n} = 0$. The proof is complete.

Eigenanalysis when A has Negative Eigenvalues. If all eigenvalues μ of A are negative or zero, then, for some $\omega \geq 0$, eigenvalue μ is related to an eigenvalue λ of C by the relation $\mu = -\omega^2 = \lambda^2$. Then $\lambda = \pm\omega i$ and $\omega = \sqrt{-\mu}$. Consider an eigenpair $(-\omega^2, \mathbf{v})$ of the real $n \times n$ matrix A with $\omega \geq 0$ and let

$$u(t) = \begin{cases} c_1 \cos \omega t + c_2 \sin \omega t & \omega > 0, \\ c_1 + c_2 t & \omega = 0. \end{cases}$$

Then $u''(t) = -\omega^2 u(t)$ (both sides are zero for $\omega = 0$). It follows that $\mathbf{x}(t) = u(t)\mathbf{v}$ satisfies $\mathbf{x}''(t) = -\omega^2 \mathbf{x}(t)$ and $A\mathbf{x}(t) = u(t)A\mathbf{v} = -\omega^2\mathbf{x}(t)$. Therefore, $\mathbf{x}(t) = u(t)\mathbf{v}$ satisfies $\mathbf{x}''(t) = A\mathbf{x}(t)$.

Theorem 34 (Eigenanalysis Solution of $\mathbf{x}'' = A\mathbf{x}$)
Let the $n \times n$ real matrix A have eigenpairs $\{(\mu_j, \mathbf{v}_j)\}_{j=1}^n$. Assume $\mu_j = -\omega_j^2$ with $\omega_j \geq 0$, $j = 1, \ldots, n$. Assume that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Then the general solution of $\mathbf{x}''(t) = A\mathbf{x}(t)$ is given in terms of $2n$ arbitrary constants $a_1, \ldots, a_n, b_1, \ldots, b_n$ by the formula

$$\mathbf{x}(t) = \sum_{j=1}^{n} \left(a_j \cos \omega_j t + b_j \frac{\sin \omega_j t}{\omega_j} \right) \mathbf{v}_j \tag{7}$$

In this expression, we use the limit convention

$$\left. \frac{\sin \omega t}{\omega} \right|_{\omega=0} = t.$$

Proof: The text preceding the theorem and superposition establish that $\mathbf{x}(t)$ is a solution. It only remains to prove that it is the general solution, meaning that the arbitrary constants can be assigned to allow any possible initial conditions $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{x}'(0) = \mathbf{y}_0$. Define the constants uniquely by the relations

$$\begin{aligned} \mathbf{x}_0 &= \textstyle\sum_{j=1}^n a_j \mathbf{v}_j, \\ \mathbf{y}_0 &= \textstyle\sum_{j=1}^n b_j \mathbf{v}_j, \end{aligned}$$

which is possible by the assumed independence of the vectors $\{\mathbf{v}_j\}_{j=1}^n$. Then (7) implies $\mathbf{x}(0) = \sum_{j=1}^n a_j \mathbf{v}_j = \mathbf{x}_0$ and $\mathbf{x}'(0) = \sum_{j=1}^n b_j \mathbf{v}_j = \mathbf{y}_0$. The proof is complete.