Sample Solution to 6.2-18

$$A = \begin{pmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

part I. College algebra

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -5 & 2 \\ 4 & -3-\lambda & 2 \\ 2 & -2 & 3-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda-1)(\lambda-2)(\lambda-3)$$

Then $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ are the eigenvalues of A (the scale factors).

part II. Linear algebra

to be solved:

$$\begin{cases} A\vec{v}_1 = (1)\vec{v}_1 \\ A\vec{v}_2 = (2)\vec{v}_2 \\ A\vec{v}_3 = (3)\vec{v}_3 \end{cases} \quad \text{or homogeneous problems} \quad \begin{cases} (A-I)\vec{v}_1 = \vec{0} \\ (A-2I)\vec{v}_2 = \vec{0} \\ (A-3I)\vec{v}_3 = \vec{0} \end{cases}$$

problem $(A-I)\vec{v}_1 = \vec{0} \longrightarrow \left(\begin{array}{ccc|c} 5 & -5 & 2 & 0 \\ 4 & -4 & 2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 4 & -4 & 2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Then $\begin{cases} x_1 - x_2 = 0 \\ x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \frac{\partial}{\partial t_1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\boxed{\text{Eigenpair } \left(1, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)}$

problem $(A-2I)\vec{v}_2 = \vec{0} \rightarrow \left(\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & \frac{1}{2} & 0 \\ 4 & -5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{cases} x_1 + \frac{1}{2}x_3 = 0 \\ x_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t_1 \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow \vec{v}_2 = \frac{\partial}{\partial t_1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$ $\quad \boxed{\text{Eigenpair } \left(2, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}\right)}$

problem $(A-3I)\vec{v}_3 = \vec{0} \longrightarrow \left(\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 4 & -6 & 2 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 3 & -5 & 2 & 0 \\ 4 & -6 & 2 & 0 \end{array}\right) \rightarrow$

$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 3 & -5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \vec{v}_3 = \frac{\partial}{\partial t_1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$ $\quad \boxed{\text{Eigenpair } \left(3, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)}$

<u>Diagonalization</u>

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & -\frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}; \text{ then } AP = PD$$