Picard–Lindelöf Theorem: The Vector Case

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Picard-Lindelöf Theorem. Let $\vec{f}(x,\vec{y})$ be defined for $|x-x_0| \le h$, $\|\vec{y}-\vec{y}_0\| \le k$, with $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ continuous. Then for some constant H, $0 < H < h$, the problem

$$\begin{cases} \vec{y}\,'(x) &= \vec{f}(x,\vec{y}(x)), \;\; |x-x_0| < H, \\ \vec{y}(x_0) &= \vec{y}_0 \end{cases}$$

has a unique solution $\vec{y}(x)$ defined on the smaller interval $|x-x_0| < H$.

Conversion of Second Order Scalar to a First Order System

Example. Transform the spring-mass system into a first order system in vector form.

$$y'' + 3y' + 2y = g(x), \quad y(0) = y_0, \quad y'(0) = y_1.$$

Let $\vec{u}(x) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}$. Then $u_1 = y(x)$, $u_2 = y'(x)$ and

$$\vec{u}'(x) = \begin{pmatrix} y'(x) \\ y''(x) \end{pmatrix} = \begin{pmatrix} y'(x) \\ g(x) - 3y'(x) - 2y(x) \end{pmatrix},$$

because of the differential equation $y'' + 3y' + 2y = g(x)$. Use $y(x) = u_1$, $y'(x) = u_2$ to write

$$\vec{u}'(x) = \begin{pmatrix} y'(x) \\ g(x) - 3y'(x) - 2y(x) \end{pmatrix} = \begin{pmatrix} u_2 \\ g(x) - 3u_2 - 2u_2 \end{pmatrix}.$$

Define $\vec{f}(x, \vec{u}) = \begin{pmatrix} u_2 \\ g(x) - 3u_2 - 2u_1 \end{pmatrix}$.

Then $\vec{u}' = \vec{f}(x, \vec{u})$ is the vector form of the spring-mass system.

The initial condition is $\vec{u}(0) = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$.

Definitions

Definition. A vector function $\vec{f}(x, \vec{u})$ is said to be continuous on a set $|x - x_0| < h$, $\|\vec{u} - \vec{u}_0\| < H$ provided for each $(x_1, \vec{u}_1)$ in the set, we have

$$\lim_{x \to x_1, \vec{u} \to \vec{u}_1} \vec{f}(x, \vec{u}) = \vec{f}(x_1, \vec{u}_1).$$

Definition. Symbol $\partial \vec{f}(x, \vec{u}) / \partial \vec{u}$ is the Jacobian matrix of partial derivatives of vector $\vec{f}$ with respect to the components of vector $\vec{u}$. If $\vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, then

$$\frac{\partial \vec{f}(x, \vec{u})}{\partial \vec{u}} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{pmatrix}$$

Jacobians and Continuity

A Jacobian matrix is said to be continuous provided all of its entries are continuous. This implies:

Theorem. A Jacobian matrix of $\vec{f}$ is continuous in variables $x, \vec{u}$ provided the partial derivatives $\partial \vec{f}(x, \vec{u})/\partial u_j$, $j = 1, \ldots, n$, are continuous in variables $x, \vec{u}$.

Example. The Jacobian matrix of $\vec{f}(x, \vec{u}) = \begin{pmatrix} u_2 \\ g(x) - 3u_2 - 2u_1 \end{pmatrix}$ is

$$J = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}.$$

The entries are polynomials, hence everywhere continuous. Therefore, $\partial \vec{f}(x, \vec{u})/\partial \vec{u}$ is continuous in variables $x, \vec{u}$.