Newton's Laws

The ideal models of a particle or *point mass* constrained to move along the x-axis, or the motion of a projectile or satellite, have been studied from **Newton's second law**

$$F = ma. \tag{1}$$

In the *mks system* of units, F is the force in **Newtons**, m is the mass in kilograms and a is the acceleration in meters per second per second.

The closely-related **Newton universal gravitation law**

$$F = G\frac{m_1 m_2}{R^2} \tag{2}$$

is used in in conjunction with (1) to determine the system's constant value g of gravitational acceleration. The masses m_1 and m_2 have centroids at a distance R. For the earth, $g = 9.8$ m/s^2 is commonly used.

Velocity and Acceleration

The position, velocity and acceleration of a particle moving along an axis are functions of time t. Notations vary; we use the following symbols, where primes denote t-differentiation.

$x = x(t)$	The particle's **position** at time t.
$v = x'(t)$	The particle's **velocity** at time t.
$a = x''(t)$	The particle's **acceleration** at time t.
$x(0)$	The **initial position**.
$v(0)$	The **initial velocity**. Synonym $x'(0)$ is also used.

Free Fall with Constant Gravity

Falling bodies, e.g., an object launched up or down from a tall building, are ideal cases, in which air resistance and other external forces are ignored. The acceleration of the body is assumed to be a constant g. The model is

$$x''(t) = -g, \quad x(0) = x_0, \quad x'(0) = v_0. \tag{3}$$

The initial position x_0 and the initial velocity v_0 must be specified. The value of g in mks units is $g = 9.8$ m/s^2. The symbol x is the distance from the ground ($x = 0$). The symbol t is the time in seconds.

Solution of the falling body problem

Equation (3) can be solved by the method of quadrature to give the explicit solution

$$x(t) = -\frac{g}{2}t^2 + x_0 + v_0 t. \tag{4}$$

Plotting

Typical plots can be made by the following `maple` code.

```
X:=unapply(-9.8*t^2+100+(50)*t,t); #v(0)=50m/s,x(0)=100m
plot(X(t),t=0..7);
```

Air Resistance Effects

The inclusion in a differential equation model of terms accounting for air resistance has historically two distinct models. The first is *linear resistance*, in which the force $\boldsymbol{F}$ due to air resistance is assumed to be proportional to the velocity $\boldsymbol{v}$:

$$\boldsymbol{F} \propto \boldsymbol{v}. \tag{5}$$

It is known that linear resistance is appropriate only for slowly moving objects. The second model is *nonlinear resistance*, modeled originally by Sir Isaac Newton himself as $\boldsymbol{F} = \boldsymbol{k}\boldsymbol{v}^2$. The literature considers a generalized nonlinear resistance assumption

$$\boldsymbol{F} \propto \boldsymbol{v}|\boldsymbol{v}|^{p} \tag{6}$$

where $0 < p \leq 1$ depends upon the *speed* of the object through the air; $p \approx 0$ is a low speed and $p \approx 1$ is a high speed. It will suffice for illustration purposes to treat just the two cases $\boldsymbol{F} \propto \boldsymbol{v}$ and $\boldsymbol{F} \propto \boldsymbol{v}|\boldsymbol{v}|$.

Linear Air Resistance

The model is determined by the sum of the forces due to air resistance and gravity, $\boldsymbol{F}_{\text{air}} + \boldsymbol{F}_{\text{gravity}}$, which by *Newton's second law* must equal $\boldsymbol{F} = mx''(t)$, giving the differential equation

$$mx''(t) = -kx'(t) - mg. \tag{7}$$

Equation (7) written in terms of the velocity $v = x'(t)$ becomes

$$v'(t) = -(k/m)v(t) - g. \tag{8}$$

This equation has a solution $v(t)$ which limits at $t = \infty$ to a **finite terminal velocity** $|v_\infty| = mg/k$,. Physically, this limit is the **equilibrium solution** of (8), which is the observable steady state of the model. A quadrature applied to $x'(t) = v(t)$ solves (7). Then

$$\begin{aligned} v(t) &= -\frac{mg}{k} + \left(v(0) + \frac{mg}{k}\right) e^{-kt/m}, \\ x(t) &= x(0) - \frac{mg}{k}t + \frac{m}{k}\left(v(0) + \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right). \end{aligned} \tag{9}$$

Nonlinear Air Resistance. The model, which applies primarily to rapidly moving objects, is obtained by the same method as the linear model, replacing the linear resistance term $kx'(t)$ by the nonlinear term $kx'(t)|x'(t)|$. The model:

$$mx''(t) = -kx'(t)|x'(t)| - mg. \tag{10}$$

The velocity $v = x'(t)$ model is

$$v'(t) = -(k/m)v(t)|v(t)| - g. \tag{11}$$

The model applies in particular to parachute flight and to certain projectile problems, like an arrow or bullet fired straight up.

Upward Launch. Separable equation (11) in the case $v(0) > 0$ for a launch upward becomes $v'(t) = -(k/m)v^2(t) - g$. The solution for $v(0) > 0$ is given below in (12). The equation $x'(t) = v(t)$ can be solved by quadrature. Then for some constants c and d

$$
\begin{aligned}
v(t) &= \sqrt{\frac{mg}{k}}\,\tan\left(\sqrt{\frac{kg}{m}}(c-t)\right), \\
x(t) &= d + \frac{m}{k}\,\ln\left|\cos\left(\sqrt{\frac{kg}{m}}(c-t)\right)\right|.
\end{aligned} \tag{12}
$$

Downward Launch. The case $v(0) < 0$ for an object launched downward or dropped will use the equation $v'(t) = (k/m)v^2(t) - g$. Then for some constants c and d

$$
\begin{aligned}
v(t) &= \sqrt{\frac{mg}{k}}\,\tanh\left(\sqrt{\frac{kg}{m}}(c-t)\right), \\
x(t) &= d - \frac{m}{k}\,\ln\left|\cosh\left(\sqrt{\frac{kg}{m}}(c-t)\right)\right|.
\end{aligned} \tag{13}
$$

The **hyperbolic functions** appearing in (13) are *defined by*

$\cosh u = \frac{1}{2}\left(e^{u} + e^{-u}\right)$	Hyperbolic cosine.
$\sinh u = \frac{1}{2}\left(e^{u} - e^{-u}\right)$	Hyperbolic sine.
$\tanh u = \dfrac{e^{u} - e^{-u}}{e^{u} + e^{-u}}$	Hyperbolic tangent. Identity $\tanh u = \sinh u / \cosh u$.

The model applies to parachute problems in particular. Equation (13) and the limit formula $\lim_{|x|\to\infty} \tanh x = 1$ imply a *terminal velocity*

$$|v_\infty| = \sqrt{\frac{mg}{k}}.$$

The value is exactly the square root of the linear model terminal velocity. Without air resistance effects, e.g., the falling body model (3), the velocity is allowed to increase to unrealistic speeds.

Details for the Hyperbolic tangent conversion.

The downward motion assumes $v(0) = 0$, because the bolt reached maximum height at $v = 0$. Because the path is downward, then $v \le 0$ and $v|v| = -v^2$, which gives a simplified model that looks like

$$\begin{cases} v' &= -1(v^2 - 1), \\ v(0) &= 0. \end{cases}$$

Separation of variables gives

$$\frac{1}{2} \ln \left| \frac{1+v}{v-1} \right| = -t + c.$$

Inverting with exponentials gives

$$-\frac{1+v}{v-1} = e^{2x}, \quad \text{where} \quad x = -t + c.$$

We erased the bars and multiplied by -1 because at $t = 0$ the fraction is $\frac{1+v}{v-1} = -1 < 0$. Solving for v:

$$v = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^{-x}}{e^{-x}} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \tanh(x), \quad \text{where} \quad x = -t + c.$$

Longer to Rise or Longer to Fall?

A bolt is shot straight upward (along the gravity vector) with initial velocity $v_1(0) = 49$ m/s from a crossbow at ground level. Assume air resistance proportional to the square of the velocity with drag $\rho = 0.0011$, giving the rise (v_1) and fall (v_2) equations

$$\frac{d}{dt}v_1(t) = -9.8 - 0.0011 v_1(t)^2,$$
$$\frac{d}{dt}v_2(t) = -9.8 + 0.0011 v_2(t)^2.$$

Compare the rise and fall times for this Newton model with the models

$$\frac{d}{dt}v_3(t) = -9.8 \text{ [no air resistance], and}$$

$$\frac{d}{dt}v_4(t) = -9.8 - 0.04 v_4(t) \text{ [linear air resistance].}$$

Longer to Rise or Longer to Fall?

```
Results
=======
Model       Force       Max       Flight   Rise   Fall   Impact
used        Equation    Height    time     time   time   velocity
=====       ========    ======    ======   ====   ====   ========
v3          F=0         122.5     10.00    5.00   5.00   49.00
v4          F=kv        108.28     9.41    4.56   4.85   43.23
v1 & v2     F=kv|v|     108.47     9.41    4.61   4.80   43.49
```

The results imply that it takes longer to fall.