

Transform Properties

Collected here are the major theorems for the manipulation of Laplace transform tables.

- Lerch's Cancellation Law
- Linearity
- The Parts Rule (t -Derivative Rule)
- The t -Integral Rule
- The s -Differentiation Rule
- First Shifting Rule
- Second Shifting Rule
- Periodic Function Rule
- Convolution Rule

Theorem 1 (Lerch)

If $f_1(t)$ and $f_2(t)$ are continuous, of exponential order and

$$\int_0^{\infty} f_1(t)e^{-st} dt = \int_0^{\infty} f_2(t)e^{-st} dt$$

for all $s > s_0$, then for $t \geq 0$,

$$f_1(t) = f_2(t).$$

The result is remembered as the cancelation law

$$L(f_1(t)) = L(f_2(t)) \text{ implies } f_1(t) = f_2(t).$$

Theorem 2 (Linearity)

The Laplace transform has these inherited integral properties:

- (a) $L(f(t) + g(t)) = L(f(t)) + L(g(t)),$
- (b) $L(cf(t)) = cL(f(t)).$

Theorem 3 (The Parts Rule)

Let $y(t)$ be continuous, of exponential order and let $y'(t)$ be piecewise continuous on $t \geq 0$. Then $L(y'(t))$ exists and

$$L(y'(t)) = sL(y(t)) - y(0).$$

Theorem 4 (The t -Integral Rule)

Let $g(t)$ be of exponential order and continuous for $t \geq 0$. Then

$$L\left(\int_0^t g(x) dx\right) = \frac{1}{s}L(g(t)).$$

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- The parts rule is also called the t -derivative rule. It is used to remove derivatives y' from Laplace equations.
 - The two rules are related by $y(t) = \int_0^t g(x) dx$.

Theorem 5 (The s -Differentiation Rule)

Let $f(t)$ be of exponential order. Then

$$L(tf(t)) = -\frac{d}{ds}L(f(t)).$$

The rule says that each factor of (t) in the integrand of a Laplace integral can be crossed out provided an operation $-d/ds$ is inserted in front of the integral. It is remembered as

multiplying by $(-t)$ differentiates the transform.

Theorem 6 (First Shifting Rule)

Let $f(t)$ be of exponential order and $-\infty < a < \infty$. Then

$$L(e^{at} f(t)) = L(f(t))|_{s \rightarrow (s-a)}.$$

The rule says that an exponential factor e^{at} in the integrand can be crossed out, provided this action is compensated by replacing s by $s - a$ in the answer. It is remembered as

multiplying by e^{at} shifts the transform $s \rightarrow s - a$.

Heaviside Step

The Step function is defined by $\text{step}(t) = 1$ for $t \geq 0$ and $\text{step}(t) = 0$ for $t < 0$. It is the same as the **unit step** $u(t)$ and the **Heaviside function** $H(t)$. Then $\text{step}(t - a)$ is the step function shifted from the origin to location $t = a$,

$$\text{step}(t - a) = \begin{cases} 1 & a \leq t < \infty, \\ \text{otherwise.} & \end{cases}$$

The function **pulse** is a finite interval step function defined by

$$\begin{aligned} \text{pulse}(t, a, b) &= \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise} \end{cases} \\ &= \text{step}(t - a) - \text{step}(t - b). \end{aligned}$$

Maple Worksheet Definitions

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step := unapply(piecewise(t >= 0, 1, 0), t);
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pulse := unapply(step(t-a) - step(t-b), (t, a, b));
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Step Function Shifting Rule

Theorem 7 (Second Shifting Rule)

Let $f(t)$ and $g(t)$ be of exponential order and assume $a \geq 0$. Let $u(t) = \text{step}(t)$. Then

$$\begin{aligned} \text{(a)} \quad & L(f(t - a)u(t - a)) = e^{-as}L(f(t)), \\ \text{(b)} \quad & L(g(t)u(t - a)) = e^{-as}L(g(t + a)). \end{aligned}$$

The relations are used to manipulate Laplace equations that arise in differential equations with piecewise defined inputs. Electrical engineering has many such examples.

Theorem 8 (Periodic Function Rule)

Let $f(t)$ be of exponential order and satisfy $f(t + P) = f(t)$. Then

$$L(f(t)) = \frac{\int_0^P f(t)e^{-st} dt}{1 - e^{-Ps}}.$$

Some Engineering Functions

Tabulated here are common periodic functions used in engineering applications.

$$L(\mathbf{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})}$$

Staircase function,
 $\mathbf{floor}(x) = \text{greatest integer } \leq x$.

$$L(\mathbf{sqw}(t/a)) = \frac{1}{s} \tanh(as/2)$$

Square wave,
 $\mathbf{sqw}(x) = (-1)^{\mathbf{floor}(x)}$.

$$L(a \mathbf{trw}(t/a)) = \frac{1}{s^2} \tanh(as/2)$$

Triangular wave,
 $\mathbf{trw}(x) = \int_0^x \mathbf{sqw}(r) dr$.

Theorem 9 (Convolution Rule)

Let $f(t)$ and $g(t)$ be of exponential order. Then

$$L(f(t))L(g(t)) = L\left(\int_0^t f(x)g(t-x)dx\right).$$

An example:

$$\begin{aligned}\frac{1}{s^2} \frac{1}{s-2} &= L(t)L(e^{2t}) \\ &= L\left(\int_0^t xe^{2(t-x)}dx\right) \\ &= L\left(\frac{1}{3}e^{2t} - \frac{1}{2}t - \frac{1}{4}\right)\end{aligned}$$