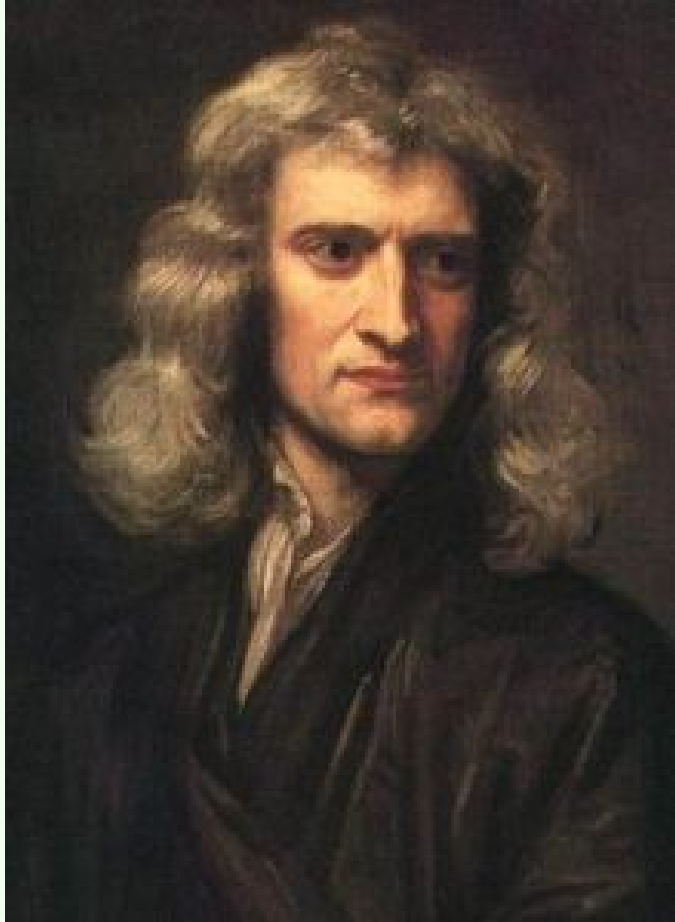# The Integral Calculus of Newton and Laplace



Isaac Newton (1643-1727) at age 46

Pierre-Simon Laplace (1749-1827)

**Abstract**

Presented here is a comparison of the integral calculus of Isaac Newton, normally learned in a university calculus course, against the Laplace calculus.

- The integral calculus of Isaac Newton is used extensively in science and engineering applications that assume a calculus background.

- The Laplace calculus is used in science and engineering to solve or model ordinary, integral and partial differential equations.

The reader is assumed to have some casual knowledge of both integral calculus and Laplace theory applications.

**The Newton Integral Calculus**

The *integral calculus* consists of rules plus an integral table.

<table>
<tr><td>**Integral Rules**</td><td>**Integral Table Entries**</td></tr>
<tr><td>Fundamental Theorem</td><td>integral of a power $x^n$</td></tr>
<tr><td>sum rule</td><td>trigonometric integrals</td></tr>
<tr><td>constant rule</td><td>exponential integrals</td></tr>
<tr><td>parts</td><td>logarithmic integrals</td></tr>
<tr><td>u-substitution</td><td>rational functions</td></tr>
<tr><td>tabular integration</td><td>inverse trigonometric integrals</td></tr>
<tr><td>trig substitution</td><td>common radicals</td></tr>
</table>

- Literature expands both the set of rules and the table.

- The objective of the calculus is to compute answers to integrals and derivatives.

## The Laplace Calculus

The *Laplace calculus* consists of rules plus an integral table, not of ordinary integrals, but the *Laplace integral* $\int_0^\infty f(t)e^{-st}dt$.

### Laplace Rules

Lerch's Theorem
sum rule
constant rule
parts
shift rule
s-derivative rule
periodic rule
convolution rule

### Laplace Integral Table

$$\int_0^\infty (t^n)e^{-st}dt = \frac{n!}{s^{n+1}}$$

$$\int_0^\infty (e^{at})e^{-st}dt = \frac{1}{s-a}$$

$$\int_0^\infty (\cos bt)e^{-st}dt = \frac{s}{s+b^2}$$

$$\int_0^\infty (\sin bt)e^{-st}dt = \frac{b}{s^2+b^2}$$

$$\int_0^\infty (\text{step}(t-a))e^{-st}dt = \frac{e^{-as}}{s}$$

- Literature expands both the set of rules and the table.

- The objective of Laplace calculus is to compute answers to Laplace integrals.

**Laplace $L$-Notation**

A Laplace integral $\int_0^\infty f(t)e^{-st}dt$ can be decomposed into

- An integral sign $\int_E$ where $E$ is the set $0 \leq t < \infty$.
- An integrand $f(t)$.
- A Laplace integrator $dx = e^{-st}dt$.

Symbol $L$ replaces the integral sign $\int_E$ and the Laplace integrator is omitted to obtain the definition

$$L(f(t)) \equiv \int_E f(t)dx = \int_0^\infty f(t)e^{-st}dt.$$

# A Comparison of Newton's Integral Calculus with Laplace Calculus _____

In the table, notation $L(f(t))$ replaces the Laplace integral $\int_0^\infty f(t)e^{-st}dt$.

| Calculus Rules | Laplace Rules | Integral Table | Laplace Table |
|---|---|---|---|
| Fundamental Theorem | Lerch's Theorem | integral of a power | $L(t^n)$ |
| sum rule | sum rule | trigonometric integrals | $L(\cos bt),\ L(\sin bt)$ |
| constant rule | constant rule | exponential integrals | $L(e^{at})$ |
| parts rule | parts rule | logarithmic integrals | |
| u-substitution | shift rules | rational functions | |
| tabular integration | s-derivative rule | inverse trig integrals | |
| trig substitution | periodic and convolution | common radicals | |

- The Newton integral table is immense, while Laplace's table is extremely small.

- Laplace rules are designed to extend the table on demand.

- Both tables can be read forward and backward. This depends on the Fundamental Theorem of Calculus for Newton's calculus and on Lerch's Theorem for the Laplace calculus.

**Quadrature Method**

The *method of quadrature* for a quadrature differential equation

$$y' = F(x)$$

requires that you *multiply by $dx$ and apply an integral sign $\int$ to each side.*

- The logic of the quadrature method is that equal integrands give equal integrals.
- The method works only because the fundamental theorem of calculus evaluates $\int y'dx = y + c$. This results in symbol $y$ being isolated on the left of the equal sign.

## Laplace's Method



When applied to scalar differential equations, integral equations, systems of differential equations and partial differential equations, Laplace's method is

- *Multiply the equation* by the Laplace integrator $dx = e^{-st}dt$ and apply an integral sign $\int_E$ to each side.

- Equivalently, *multiply the equation* by $L$, as though $L$ was a matrix.

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- Any derivatives are eliminated by the parts rule $L(y') = sL(y) - y(0)$. Then only $L(y)$ appears in the equation(s).

- Because $y$ does not explicitly appear in the equation, but $L(y)$ does, then Lerch's theorem must be used to find $y$, by backwards table lookup.