Shortcuts for Solving First Order Linear Differential Equations
$$y' + p(x)y = r(x)$$

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Throughout these slides, the *integrating factor* for $y' + p(x)y = r(x)$ is the expression

$$W(x) = e^{u(x)}, \quad u(x) = \int p(x)\,dx.$$

Theorem 1 (Homogeneous Equation Shortcut)
Assume $p(x)$ is continuous on an interval. Then the solution of the homogeneous differential equation $y' + p(x)y = 0$ on that interval is given by the formula

$$y(x) = \frac{\text{constant}}{\text{integrating factor}} = \frac{c}{W(x)}. \tag{1}$$

Proof: Replace the left side $y' + p(x)y$ of the homogeneous equation by the integrating factor quotient $(yW)'/W$. Cross-multiply to get $(yW)' = 0$. Integrate: $yW = c$. Then $y = \frac{c}{W}$, as claimed.

Theorem 2 (Constant-Coefficient Shortcut)
Assume A and B are constants with $A \neq 0$. Differential equation $y' + Ay = B$ has general solution

$$y(x) = \frac{\text{constant}}{\text{integrating factor}} + \text{equilibrium solution} = \frac{c}{e^{Ax}} + \frac{B}{A}. \tag{2}$$

Proof: The homogeneous solution is a constant divided by the integrating factor, by Theorem 1. An equilibrium solution can be found by formally setting $y' = 0$, then solving for $y = B/A$. By the superposition Theorem, the solution y must be the sum of these two solutions. We remark that the case $A = 0$ results in a quadrature equation $y' = B$ which is routinely solved by the method of quadrature.

Theorem 3 (Particular Solution Shortcut)
Assume $p(x)$ and $r(x)$ are continuous functions on an interval. Differential equation $y' + p(x)y = r(x)$ on this interval has a particular solution

$$
\begin{aligned}
y_p(x) &= \frac{\text{integral of } r(x) \text{ times an integrating factor}}{\text{integrating factor}} \\
&= \frac{\int r(x)W(x)\,dx}{W(x)}, \quad W(x) = e^{\int p(x)dx}.
\end{aligned} \tag{3}
$$

Proof: Replace $y' + p(x)y$ by the integrating factor quotient $(yW)'/W$, then $(yW)' = rW$. Integrate both sides of this last equation: $yW = \int rW\,dx + c$. Choose $c = 0$ to obtain the particular solution $y_p(x)$ reported in equation (3).

Linear Integrating Factor Shortcut

Theorem 4
(Superposition Principle) The general solution of $y' + p(x)y = r(x)$ is given by

$$y = y_h + y_p$$

where y_h is the general solution of the **homogeneous equation** $y' + p(x)y = 0$ and y_p is a particular solution of the **non-homogeneous equation** $y' + p(x)y = r(x)$.

A **shortcut** to the linear integrating factor method is superposition plus two shortcuts:

$$y(x) = y_h(x) + y_p(x),$$

$$y_h(x) = \frac{\text{constant}}{\text{integrating factor}},$$

$$y_p(x) = \frac{\text{integral of } r \text{ times integrating factor}}{\text{integrating factor}},$$

$$\text{integrating factor} = W(x) = e^{\int p(x)dx}.$$

1 Example (Shortcut: Homogeneous Equation)

(a) Solve the homogeneous equation $2y' + x^2y = 0$.

(b) Solve the homogeneous equation $(x+1)y' + xy = 0$.

Solution:

(a) By Theorem 1, the solution is a constant divided by the integrating factor. First, divide by 2 to get $y' + p(x)y = 0$ with $p(x) = \frac{1}{2}x^2$. Then $\int p(x)dx = x^3/6 + c$ implies $W = e^{x^3/6}$ is an integrating factor. The solution is $y = \dfrac{c}{e^{x^3/6}}$.

(b) By Theorem 1, the solution is a constant divided by the integrating factor.

Homogeneous Form $y' + py = 0$. Divide by $x+1$ to get $y' + p(x)y = 0$ with $p(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1}$.

Integrating Factor W. Evaluate $\int p(x)dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \ln|x+1| + c$. Then $W = e^{x - \ln|x+1|}$ is an integrating factor. We use rules $e^{\ln u} = u$ for $u > 0$, $b \ln u = \ln\left(u^b\right)$ and $e^{a+b} = e^a e^b$. Then $W = e^x e^{-\ln|x+1|} = e^x e^{\ln\left(|x+1|^{-1}\right)} = e^x|x+1|^{-1} = \frac{e^x}{(\pm 1)(x+1)}$. Due to the common factor of ± 1, another integrating factor is $W = \frac{e^x}{x+1}$.

Shortcut. The solution is $y = \frac{c}{W} = \dfrac{c(x+1)}{e^x}$.

2 Example (Shortcut: Constant-Coefficient Equation)

Solve the non-homogeneous constant-coefficient equation $2y' - 5y = -\sqrt{\pi}$.

Solution: The method described here only works for first order constant-coefficient differential equations. If $y' = f(x, y)$ is not linear or it fails to have constant coefficients, then the method fails.
The solution has two steps:

(1) Find the solution y_h of the homogeneous equation $2y' - 5y = 0$.
The answer is a constant divided by the integrating factor.

Standard Form. First divide the equation by 2 to obtain the standard form $y' + (-5/2)y = 0$. Identify $p(x) = -5/2$.
Integrating Factor. Integrate: $\int p(x)dx = -5x/2 + c$. Then $W = e^{-5x/2}$ is the integrating factor.
Shortcut. The answer is $y_h = \dfrac{c}{W} = c\,e^{5x/2}$.

(2) Find an equilibrium solution y_p for $2y' - 5y = -\sqrt{\pi}$.
This answer is found by formally replacing y' by zero. Then $2(0) - 5y = -\sqrt{\pi}$ implies $y_p = \frac{\sqrt{\pi}}{5}$.

The general solution is the sum of the answers from (1) and (2), by superposition, giving

$$y = y_h + y_p = ce^{5x/2} + \frac{\sqrt{\pi}}{5}.$$

The method of this example might be called the **superposition method shortcut**.

3 Example (Particular Solution Shortcut)

Find a particular solution of non-homogeneous equation $xy' + y = x^2$.

Solution: The answer is $y_p(x) = \frac{x^3/3}{x} = \frac{x^2}{3}$. The shortcut has these steps:

Standard Form. Divide by x: $y' + (1/x)y = x$. Then $p = 1/x$, $r = x$.
Homogeneous Equation. Replace the RHS by zero, $y' + (1/x)y = 0$. Then $p = 1/x$.
Integrating Factor W. Evaluate $\int p(x)dx = \int(1/x)dx = \ln|x| + c_1$. We drop c_1 to simplify. Then $W = e^{\ln|x|}$ is an integrating factor. Because $e^{\ln u} = u$, then $W = |x| = (\pm 1)x$. We drop the factor ± 1 to obtain simplified integrating factor $W = x$.
Particular Solution Shortcut. First integrate $\int rW\,dx = \int x\,x\,dx = \frac{x^3}{3} + c_2$. Drop constant c_2. Then y_p is this answer divided by W: $y_p = \frac{\frac{x^3}{3}}{x} = \frac{x^2}{3}$.

4 Example (Superposition Shortcut)

Solve by the superposition shortcut the non-homogeneous equation $xy' + y = x^2$.

Solution: The solution has two steps:

(1) Find the solution y_h of the homogeneous equation $xy' + y = 0$.

The homogeneous solution shortcut details are in the preceding example: $y_h = \frac{c}{W} = \frac{c}{x}$.

(2) Find a particular solution y_p for $xy' + y = x^2$.

The steps for the particular solution shortcut are those of the preceding example: $y_p = \frac{x^2}{3}$.

The general solution is the sum of the answers from (1) and (2), by superposition, giving

$$y = y_h + y_p = \frac{c}{x} + \frac{x^2}{3}.$$