The Integrating Factor Method for a Linear Differential Equation $y' + p(x)y = r(x)$

Contents

Superposition

Consider the homogeneous equation

$$y' + p(x)y = 0 \tag{1}$$

and the non-homogeneous equation

$$y' + p(x)y = r(x) \tag{2}$$

where p and r are continuous in an interval J.

Theorem 1 (Superposition)
The general solution of the non-homogeneous equation (2) is given by

$$y = y_h + y_p$$

where y_h is the general solution of homogeneous equation (1) and y_p is a particular solution of non-homogeneous equation (2).

Integrating Factor Identity

The technique called the **integrating factor method** uses the replacement rule

(3) $$\text{Fraction } \frac{(YW)'}{W} \text{ replaces } Y' + p(x)Y, \text{ where } W = e^{\int p(x)dx}.$$

The factor $W = e^{\int p(x)dx}$ in (3) is called an **integrating factor**. The fraction $\frac{(YW)'}{W}$ is called the **Integrating Factor Quotient**.

Details

Let $W = e^{\int p(x)dx}$. Then $W' = pW$, by the rule $(e^x)' = e^x$, the chain rule and the fundamental theorem of calculus $(\int p(x)dx)' = p(x)$.

Let's prove $(WY)'/W = Y' + pY$. The derivative product rule implies

$$\begin{aligned}(YW)' &= Y'W + YW' \\ &= Y'W + YpW \\ &= (Y' + pY)W.\end{aligned}$$

Divide by W. The proof is complete.

The Integrating Factor Method

Standard Form	Rewrite $y' = f(x,y)$ in the form $y' + p(x)y = r(x)$ where p, r are continuous. The method applies only in case this is possible.
Find W	Find a simplified formula for $W = e^{\int p(x)dx}$. The antiderivative $\int p(x)dx$ can be chosen conveniently (set integration constants to zero).
Prepare for Quadrature	Obtain the new equation $\frac{(yW)'}{W} = r$ by replacing the left side of $y' + p(x)y = r(x)$ by the *Integrating Factor Quotient*, equivalence (3).
Method of Quadrature	Clear fractions to obtain $(yW)' = rW$. Apply the method of quadrature to get $yW = \int r(x)W(x)dx + C$. Divide by W to isolate the explicit solution $y(x)$.

Equation (3) is central to the method, because it collapses the two terms $y' + py$ into a single term $(yW)'/W$; the method of quadrature applies to $(yW)' = rW$. Literature calls the exponential factor W an **integrating factor** and equivalence (3) a **factorization** of $y' + p(x)y$.

Integrating Factor Example 1

Example. Solve the linear differential equation $xy' + y = x^2$.

Solution: The standard form of the linear equation is

$$y' + \frac{1}{x}y = x.$$

Let

$$W = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x, \quad x > 0,$$

and replace the LHS of the differential equation by $(yW)'/W$ to obtain the quadrature equation

$$(yW)' = xW \text{ equivalent to } (yx)' = x^2.$$

Apply quadrature to this equation, then divide by W. The answer is

$$y = \frac{x^2}{3} + \frac{c}{x}.$$

Integrating Factor Example 2

Example. Solve the linear differential equation $(x+1)y' - y = -1$.

Solution: The answer is $y = 1 + c(x+1)$.
The standard form $y' + p(x)y = r(x)$ of the linear equation is

$$y' + \left(\frac{-1}{x+1}\right) y = \frac{-1}{x+1}, \quad p(x) = \frac{-1}{x+1}, \quad r(x) = \frac{-1}{x+1}.$$

Let

$$W = e^{\int \frac{-1}{x+1} dx} = e^{-\ln|x+1|} = e^{\ln\left(\frac{1}{|x+1|}\right)} = \frac{1}{|x+1|},$$

where integration constants have been set to zero. Because $|x+1| = (\pm 1)(x+1)$, then the factor ± 1 can be dropped to give a simplified integrating factor $W = \frac{1}{x+1}$. Replace the LHS of the differential equation by the *Integrating Factor Quotient* $(yW)'/W$ to get a quadrature equation

$$(yW)' = \frac{-1}{x+1} W \quad \text{equivalent to} \quad (y/(x+1))' = \frac{-1}{(x+1)^2}.$$

Quadrature gives $yW = \frac{1}{x+1} + c$. Divide by $W = \frac{1}{x+1}$, then $y = 1 + c(x+1)$.

Variation of Parameters

The initial value problem

$$y' + p(x)y = r(x), \quad y(x_0) = 0, \tag{4}$$

where p and r are continuous in an interval containing $x = x_0$, has a particular solution

$$y(x) = e^{-\int_{x_0}^{x} p(s)ds} \int_{x_0}^{x} r(t)e^{\int_{x_0}^{t} p(s)ds}\, dt. \tag{5}$$

Formula (5) is called **variation of parameters**, for historical reasons.

The formula determines a particular solution y_p which can be used in the superposition identity $y = y_h + y_p$.

While (5) has some appeal, applications use the **integrating factor method**, which is developed with indefinite integrals for computational efficiency. No one memorizes (5); they remember and study the *Integrating Factor Method*.

The **Particular Solution Shortcut** summarizes the variation of parameters formula without selecting integration constant zero. The shortcut in terms of the integrating factor $W = e^{\int p(x)dx}$ is:

$$y_p(x) = \frac{\int r(x)W(x)dx}{W(x)}.$$