Chapter 1

Fundamentals

Contents

Introduced here are notation, definitions and background results suitable for use in differential equations.

Prerequisites include college algebra, coordinate geometry, differential calculus and integral calculus. The examples and exercises include a review of some calculus topics, especially derivatives, integrals, numerical integration, hand and computer graphing. A significant part of the review is algebraic manipulation of logarithms, exponentials, sines and cosines.

New topics of an elementary nature are introduced. The chapter starts immediately with applications to differential equations that require only a background from pre-calculus in exponential and logarithmic functions. No differential equations background is assumed or used.

Differential equations are defined and insight is given into the notion of *answer* for differential equations in science and engineering.

Basic topics included here are direction fields, phase line diagrams and bifurcation diagrams, which require only a calculus background. Applications of these ideas appear later in the text, after more solution methods have been introduced.

Advanced topics include existence-uniqueness theory and implicit functions. Included are some practical methods for employing computer algebra systems to assist with finding solutions, verifying equations, modeling, and related topics.

1.1 Exponential Modeling

The model differential equation $y' = ky$ is studied through a variety of specific applications. All applications use the exponential solution $y = y_0 e^{kt}$.

Three Examples

These applications are studied:

> Growth–Decay Models
> Newton Cooling
> Verhulst Logistic Model

It is possible to solve a variety of differential equations without reading this book or any other differential equations text. Given in the table below are three exponential models and their known solutions, all of which will be derived from principles of elementary differential calculus.

Growth-Decay
$$\frac{dA}{dt} = kA(t),\ A(0) = A_0$$

$$\boxed{A(t) = A_0 e^{kt}}$$

Newton Cooling
$$\frac{du}{dt} = -h(u(t) - u_1),\ u(0) = u_0$$

$$\boxed{u(t) = u_1 + (u_0 - u_1)e^{-ht}}$$

Verhulst Logistic
$$\frac{dP}{dt} = (a - bP(t))P(t),\ P(0) = P_0$$

$$\boxed{P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}}$$

These models and their solution formulas form a foundation of intuition for all of differential equation theory. Considerable use will be made of the models and their solution formulas.

The *physical meanings* of the constants k, A_0, h, u_1, u_0, a, b, P_o and the variable names $A(t)$, $u(t)$, $P(t)$ are given below, as each example is discussed.

Background

Mathematical background used in exponential modeling is limited to algebra and basic calculus. The following facts are assembled for use in applications. Prime notation is used: $' = \frac{d}{dt}$.

$\ln e^x = x, \; e^{\ln y} = y$	In words, the exponential and the logarithm are inverses. The domains are $-\infty < x < \infty$, $0 < y < \infty$.
$e^0 = 1, \; \ln(1) = 0$	Special values, usually memorized.
$e^{a+b} = e^a e^b$	In words, the exponential of a sum of terms is the product of the exponentials of the terms.
$(e^a)^b = e^{ab}$	Negatives are allowed, e.g., $(e^a)^{-1} = e^{-a}$.
$\left(e^{u(t)}\right)' = u'(t) e^{u(t)}$	The *chain rule* of calculus implies this formula from the identity $\frac{d}{dx} e^x = e^x$. Notation $' = \frac{d}{dx}$.
$\ln AB = \ln A + \ln B$	In words, the logarithm of a product of factors is the sum of the logarithms of the factors.
$B \ln(A) = \ln\left(A^B\right)$	Negatives are allowed, e.g., $-\ln A = \ln(1/A)$ when $B = -1$.
$(\ln|u(t)|)' = \dfrac{u'(t)}{u(t)}$	The identity $\frac{d}{dx} \ln(x) = 1/x$ implies this general version by the *chain rule*.

Applied topics using exponentials inevitably lead to equations involving logarithms. Conversion of exponential equations to logarithmic equations, and the reverse, happens to be an important subtopic of differential equations. The examples and exercises contain typical calculations.

Growth-Decay Model

Growth and decay models in science are based upon the exponential equation

$$(1) \qquad\qquad y = y_0 e^{kx}, \quad y_0 \text{ and } k \text{ constant.}$$

The exponential e^{kx} increases if $k > 0$ and decreases if $k < 0$. A model based upon the exponential is called a **growth model** if $k > 0$ and a **decay model** if $k < 0$. Examples of growth models include population growth and compound interest. Examples of decay models include radioactive decay, radiocarbon dating and drug elimination. Typical growth and decay curves appear in Figure 1.
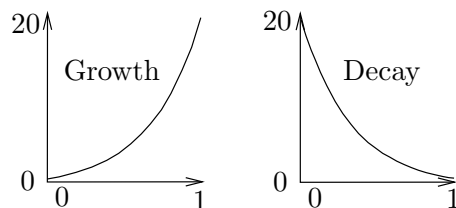
Figure 1. Growth and decay curves.

Definition 1 (Growth-Decay Equation)
The differential equation

(2)
$$\frac{dy}{dx} = ky$$

is called a **growth-decay** differential equation.

A solution of (2) is $y(x) = y_0 e^{kx}$; see the verification on page 10. It is possible to show directly that the differential equation has no other solutions, hence the terminology *the solution* $y = y_0 e^{kx}$ is appropriate; see the verification on page 11. The solution $y = y_0 e^{kx}$ in (1) satisfies the **growth-decay initial value problem**

(3)
$$\frac{dy}{dx} = ky, \quad y(0) = y_0.$$

The **initial condition** $y(0) = y_0$ means $y = y_0$ at $x = 0$. It can be written as $y(x)|_{x=0} = y_0$.

How to Solve a Growth-Decay Equation.   Numerous applications to first order differential equations are based upon equations that have the general form $\frac{dy}{dx} = ky$. Whenever this form is encountered, immediately the solution is known: $y(x) = y_0 e^{kx}$. The symbol y_0 is a constant known as the **initial state**, because $e^{kx} = 1$ at $x = 0$ implies $y(x)$ equals y_0 at $x = 0$.

Newton Cooling Model

If a fluid is held at constant temperature, then the cooling of a body immersed in the fluid is subject to **Newton's cooling law**:

> The rate of temperature change of the body is proportional to the difference between the body's temperature and the fluid's constant temperature.

Translation to mathematical notation gives the differential equation

(4)
$$\frac{du}{dt} = -h(u(t) - u_1)$$

where $u(t)$ is the temperature of the body, u_1 is the constant ambient temperature of the fluid and $h > 0$ is a constant of proportionality.

A typical instance is the cooling of a cup of hot chocolate in a room. Here, u_1 is the wall thermometer reading and $u(t)$ is the reading of a dial thermometer immersed in the chocolate drink.

Theorem 1 (Solution of Newton's Cooling Equation)

The change of variable $y(t) = u(t) - u_1$ translates the cooling equation $du/dt = -h(u - u_1)$ into the growth-decay equation $\dfrac{d}{dt}y(t) = -hy(t)$. Therefore, the cooling solution is given in terms of $u_0 = u(0)$ by the equation

$$(5) \qquad\qquad u(t) = u_1 + (u_0 - u_1)e^{-ht}.$$

The result is proved on page 11. It shows that a cooling model is just a translated growth-decay model. The solution formula (5) can be expressed in words as follows:

> The dial thermometer reading of the hot chocolate equals the wall thermometer reading plus an exponential decay term.

Cooling problems have *curious extra conditions*, usually involving physical measurements, for example the three equations

$$u(0) = 100, \quad u(1) = 90 \quad \text{and} \quad u(\infty) = 22.$$

The extra conditions implicitly determine the actual values of the three undetermined parameters h, u_1, u_0. The logic is as follows. Equation (5) is a relation among 5 variables. Substitution of values for t and u eliminates 2 of the 5 variables and gives an equation for u_1, u_0, h. The system of three equations in three unknowns can be solved for the actual values of u_1, u_0, h.

Stirring Effects. Exactly how to maintain a constant ambient temperature is not addressed by the model. One method is to stir the liquid, as in Figure 2, but the mechanical energy of the stirrer will inevitably appear as heat in the liquid. In the simplest case, stirring effects add a fixed constant temperature S_0 to the model. For slow stirring, $S_0 = 0$ is assumed, which is the above model.
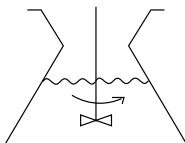

Figure 2. Flask Cooling with Stirring.

Population Modeling

The **human population** of the world reached six billion in 1999, according to the U.S. Census Bureau.

World Population Estimate
12/1999

6,033,366,287

Source: U.S. Census Bureau

The term **population** refers to humans. In literature, it may also refer to bacteria, insects, rodents, rabbits, wolves, trees, yeast and similar living things that have birth rates and death rates.

Malthusian Population Model. A constant birth rate or a constant death rate is unusual in a population, but these ideal cases have been studied. The biological reproduction law is called **Malthus's law**:

> The population flux is proportional to the population itself.

This biological law can be written in calculus terms as

$$\frac{dP}{dt} = kP(t)$$

where $P(t)$ is the population count at time t. The reasoning is that **population flux** is the expected change in population size for a unit change in t, or in the limit, dP/dt. A careful derivation of such calculus laws from English language appears in Example 6 on page 874.

The theory of growth-decay differential equations implies that population studies based upon Malthus's law employ the exponential model

$$P(t) = P_0 e^{k(t-t_0)}.$$

The number k is the difference of the birth and death rates, or **combined birth-death rate**, t_0 is the **initial time** and P_0 is the **initial population size** at time $t = t_0$.

Verhulst Logistic Model. The population model $P' = kP$ was studied around 1840 by the Belgian demographer and mathematician Pierre-Francois Verhulst (1804–1849) in the special case when k depends on the population size $P(t)$. Under Verhulst's assumptions, $k = a - bP$ for positive constants a and b, so that $k > 0$ (growth) for populations smaller than a/b and $k < 0$ (decay) when the population exceeds a/b. The result is called the **logistic equation**:

(6)
$$P' = (a - bP)P.$$

Verhulst established the limit formula

(7)
$$\lim_{t \to \infty} P(t) = a/b,$$

which has the interpretation that initial populations $P(0)$, regardless of size, will after a long time stabilize to size approximately a/b. The constant a/b is called the **carrying capacity** of the population.

Limit formula (7) follows directly from solution formula (8) below.

Theorem 2 (Verhulst Logistic Solution)
The change of variable $y(t) = P(t)/(a - bP(t))$ transforms the logistic equation $P'(t) = (a - bP(t))P(t)$ into the growth-decay equation $y'(t) = ay(t)$. Then the logistic equation solution is given by

$$(8) \qquad P(t) = \frac{aP(0)}{bP(0) + (a - bP(0))e^{-at}}.$$

The derivation appears on page 11. The impact of the result is that a logistic model transforms to a growth-decay model via a fractional change of variable. The **Verhult logistic model** reduces to the **Malthus model** when $b = 0$. Then solution formula (8) reduces to the solution $y = y_0 e^{at}$ of growth-decay equation $y' = ay$, where $y = P, y_0 = P(0)$. Solution formula (8) remains valid regardless of the signs of a and b, provided the quotient is defined.

Examples

1 Example (Growth-Decay) Solve the initial value problem

$$\frac{dy}{dx} = 2y, \quad y(0) = 4.$$

Solution: This is a growth-decay equation $y' = ky$, $y(0) = y_0$ with $k = 2$, $y_0 = 4$. One way to decide on the constant k is to compute y'/y from the given differential equation. Then $y'/y = 2$ implies $k = 2$. Therefore, the solution is $y = y_0 e^{kx} = 4e^{2x}$. No method is required to *solve* the equation $y' = 2y$, because of the theory on page 3.

2 Example (Newton Cooling) Solve the initial value problem

$$\frac{du}{dt} = -3(u(t) - 72), \quad u(0) = 190.$$

Solution: This is a Newton cooling equation $u' = -h(u - u_1)$, $u(0) = u_0$ with $h = 3$, $u_1 = 72$, $u_0 = 190$. Therefore, the solution is $u(t) = u_1 + (u_0 - u_1)e^{-ht} = 72 + 118e^{-3t}$. No method is required to *solve* the equation $u' = -3(u - 72)$, because of the theorem on page 5.

To reduce memorization, use the substitution $y = u - u_1$ to transform the problem to the growth-decay model $y' = -hy$ with solution $y = y_0 e^{-ht}$. Then back-substitute $y = u - u_1$ to solve for $u(t)$.

In this particular case, let $y = u - 72$ to get $y' = -3y$, then $y = y_0 e^{-3t}$ and finally $u - 72 = y_0 e^{-3t}$. The value of y_0 is determined by the condition $u = 4$ at $t = 0$ (supplied as $u(0) = 4$) to give $y_0 = 118$ and then the final answer is $u(t) = 72 + 118e^{-3t}$.

3 Example (Verhulst Logistic) Solve the initial value problem

$$\frac{dP}{dt} = (1 - 2P)P, \quad P(0) = 500.$$

Solution: This is a Verhulst logistic equation $P' = (a - bP)P$, $P(0) = P_0$ with $a = 1$, $b = 2$, $P_0 = 500$. Therefore, the solution is

$$P(t) = \frac{500}{1000 - 999e^{-t}}.$$

No method is required to *solve* the equation $P' = (1 - 2P)P$, because of the formula supplied by Theorem 2.

Because of Verhulst solution formula complexity, there is no practical shortcut to obtain the solution. The easiest route is to use the solution formula in Theorem 2.

4 Example (Standing Room Only) Justify the estimate 2600 for the year in which each human has only one square foot of land to stand upon. Assume the Malthus model $P(t) = 3.34e^{0.02(t-1965)}$, with t in years and P in billions.

Solution: The mean radius of the earth is 3965 miles or $20,935,200$ feet. The surface area formula $4\pi r^2$ gives $5,507,622$ billion square feet. About 20% of this is land, or $1,101,524$ billion square feet.

The estimate 2600 is obtained by solving for t years in the equation

$$3.34e^{0.02(t-1965)} = 1101524.$$

The college algebra details:

$e^{0.02(t-1965)} = \dfrac{1101524}{3.34}$ Isolate the exponential on the left. Solving for t.

$\ln e^{0.02(t-1965)} = \ln 329797.6$ Simplify the right side and take the logarithm of both sides.

$0.02(t - 1965) = 12.706234$ On the right, compute the logarithm. Use $\ln e^u = u$ on the left.

$t = 1965 + \dfrac{12.706234}{0.02}$ Solve for t.

$= 2600.3.$ About the year 2600.

5 Example (Rodent Growth) A population of two rodents in January reproduces to population sizes 20 and 110 in June and October, respectively. Determine a Malthusian law for the population and test it against the data.

Solution: However artificial this example might seem, it is almost a real experiment; see Braun [**?**], Chapter 1, and the reference to rodent *Microtus Arvallis Pall.*

The law proposed is $P = 2e^{2t/5}$, which is 40% growth, $k = 2/5$. For a 40% rate, $P(6) \approx 2e^{12/5} = 22.046353$ and $P(10) \approx 2e^{2(10)/5} = 109.1963$. The agreement with the data is reasonable. It remains to explain how this "40% law" was invented.

The Malthusian model $P(t) = P_0 e^{kt}$, with t in months, fits the three data items $P(0) = 2$, $P(6) = 20$ and $P(10) = 110$ provided $P_0 = 2$, $2e^{6k} = 20$ and $2e^{10k} = 110$. The exponential equations are solved for $k = \ln(10)/6$ and $k = \ln(55)/10$, resulting in the two growth constants $k = 0.38376418$ and $k = 0.40073332$. The average growth rate is 39.2%, or about 40%.

6 Example (Flask Cooling) A flask of water is heated to 95C and then allowed to cool in ambient room temperature 21C. The water cools to 80C in three minutes. Verify the estimate of 48 minutes to reach 23C.

Solution: Basic modeling by Newton's law of cooling gives the temperature as $u(t) = u_1 + (u_0 - u_1)e^{-kt}$ where u_1, u_0 and k are parameters. Three conditions are given in the English statement of the problem.

$$u(\infty) = 21 \qquad \text{The ambient air temperature is 21C.}$$
$$u(0) = 95 \qquad \text{The flask is heated at } t = 0 \text{ to 95C.}$$
$$u(3) = 80 \qquad \text{The flask cools to 80C in three minutes.}$$

In the details below, it will be shown that the parameter values are $u_1 = 21$, $u_0 - u_1 = 74$, $k = 0.075509216$.

To find u_1:

$$21 = u(\infty) \qquad \text{Given ambient temperature condition.}$$
$$= \lim_{t \to \infty} u(t) \qquad \text{Definition of } u(\infty).$$
$$= \lim_{t \to \infty} u_1 + (u_0 - u_1)e^{-kt} \qquad \text{Definition of } u(t).$$
$$= u_1 \qquad \text{The exponential has limit zero.}$$

To calculate $u_0 - u_1 = 74$ from $u(0) = 95$:

$$95 = u(0) \qquad \text{Given initial temperature condition.}$$
$$= u_1 + (u_0 - u_1)e^{-k(0)} \qquad \text{Definition of } u(t) \text{ at } t = 0.$$
$$= 21 + u_0 - u_1 \qquad \text{Use } e^0 = 1.$$

Therefore, $u_0 - u_1 = 95 - 21 = 74$.

Computation of k starts with the equation $u(3) = 80$, which reduces to $21 + 74e^{-3k} = 80$. This exponential equation is solved for k as follows:

$$e^{-3k} = \frac{80 - 21}{74} \qquad \text{Isolate the exponential factor on the left side of the equation.}$$

$$\ln e^{-3k} = \ln \frac{80 - 21}{74} \qquad \text{Take the logarithm of both sides.}$$

$$-3k = \ln(59/74)$$
Simplify the fraction. Apply $\ln e^u = u$ on the left.

$$k = \frac{1}{3} \ln(74/59)$$
Divide by -3, then on the right use $-\ln x = \ln(1/x)$.

The estimate $u(48) \approx 23$ will be verified. The time t at which $u(t) = 23$ is found by solving the equation $21 + 74e^{-kt} = 23$ for t. A checkpoint is $-kt = \ln(2/74)$, from which t is isolated on the left. After substitution of $k = 0.075509216$, the value is $t = 47.82089$.

7 Example (Baking a Roast) A beef roast at room temperature 70F is put into a 350F oven. A meat thermometer reads 100F after four minutes. Verify that the roast is done $(340F)$ in 120 minutes.

Solution: The roast is done when the thermometer reads 340F or higher. If $u(t)$ is the meat thermometer reading after t minutes, then it must be verified that $u(120) \geq 340$.

Even though the roast is heating instead of cooling, the beef roast temperature $u(t)$ after t minutes is given by the Newton cooling equation $u(t) = u_1 + (u_0 - u_1)e^{-kt}$, where u_1, u_0 and k are parameters. Three conditions appear in the statement of the problem:

$$u(\infty) = 350$$ The ambient oven temperature is 350F.

$$u(0) = 70$$ The beef is 70F at $t = 0$.

$$u(4) = 100$$ The roast heats to 100F in four minutes.

As in the *flask cooling example*, page 9, the first two relations above lead to $u_1 = 350$ and $u_0 - u_1 = -280$. The last relation determines k from the equation $350 - 280e^{-4k} = 100$. Solving by the methods of the flask cooling example gives $k = \frac{1}{4} \ln(280/250) \approx 0.028332171$. Then $u(120) = 350 - 280e^{-120k} \approx 340.65418$.

Details and Proofs

Growth-Decay Equation Existence Proof. It will be verified that $y = y_0 e^{kx}$ is a solution of $y' = ky$. It suffices to expand the left side (LHS) and right side (RHS) of the differential equation and compare them for equality.

$$\text{LHS} = \frac{dy}{dx}$$ The left side of $\frac{dy}{dx} = ky$ is dy/dx.

$$= \frac{d}{dx}\left(y_0 e^{kx}\right)$$ Substitute $y = y_0 e^{kx}$.

$$= y_0 k e^{kx}$$ Apply the rule $(e^u)' = u' e^u$.

$$\text{RHS} = ky$$ The right side of $\frac{dy}{dx} = ky$ is ky.

$$= k(y_0 e^{kx})$$ Substitute $y = y_0 e^{kx}$.

Therefore, LHS = RHS. This completes the proof.

Growth-Decay Equation Uniqueness Proof. It will be shown that $y = y_0 e^{kx}$ is the only solution of $y' = ky$, $y(0) = y_0$. The idea is to reduce the question to the application of a result from calculus. This is done by a clever change of variables, which has been traced back to Kümmer.[1]

Assume that y is a given solution of $y' = ky$, $y(0) = y_0$. It has to be shown that $y = y_0 e^{kx}$.

Define $v = y(x)e^{-kx}$. This defines a change of variable from y into v. Then

$$
\begin{aligned}
v' &= (e^{-kx}y)' & &\text{Compute } v' \text{ from } v = e^{-kx}y. \\
&= -ke^{-kx}y + e^{-kx}y' & &\text{Apply the product rule } (uy)' = u'y + uy'. \\
&= -ke^{-kx}y + e^{-kx}(ky) & &\text{Use the differential equation } y' = ky. \\
&= 0. & &\text{The terms cancel.}
\end{aligned}
$$

In summary, $v' = 0$ for all x. The calculus result to be applied is:

> The only function $v(x)$ that satisfies $v'(x) = 0$ on an interval is $v(x) = $ constant.

The conclusion is $v(x) = v_0$ for some constant v_0. Then $v = e^{-kx}y$ gives $y = v_0 e^{kx}$. Setting $x = 0$ implies $v_0 = y_0$ and finally $y = y_0 e^{kx}$. This completes the verification.

Newton Cooling Solution Verification (Theorem 1). The substitution $A(t) = u(t) - u_1$ will be applied to find an equivalent growth-decay equation:

$$
\begin{aligned}
\frac{dA}{dt} &= \frac{d}{dt}\left(u(t) - u_1\right) & &\text{Definition of } A = u - u_1. \\[2mm]
&= u'(t) - 0 & &\text{Derivative rules applied.} \\[2mm]
&= -h(u(t) - u_1) & &\text{Cooling differential equation applied.} \\[2mm]
&= -hA(t) & &\text{Definition of } A.
\end{aligned}
$$

The conclusion is that $A'(t) = -hA(t)$. Then $A(t) = A_0 e^{-ht}$, from the theory of growth-decay equations. The substitution gives $u(t) - u_1 = A_0 e^{-ht}$, which is equivalent to equation (5), provided $A_0 = u_0 - u_1$. The proof is complete.

Logistic Solution Verification (Theorem 2). Given $a > 0$, $b > 0$ and the logistic equation $P' = (a - bP)P$, the plan is to derive the solution formula

$$
P(t) = \frac{aP(0)e^{at}}{bP(0)e^{at} + a - bP(0)}.
$$

Assume $P(t)$ satisfies the logistic equation. Suppose it has been shown (see below) that the variable $u = P/(a - bP)$ satisfies $u' = au$. By the exponential theory, $u = u_0 e^{at}$, hence

$$
P = \frac{au}{1 + bu} \qquad\qquad \text{Solve } u = P/(a - bP) \text{ for } P \text{ in terms of } u.
$$

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[1]The German mathematician E. E. Kümmer, in his paper in 1834, republished in 1887 in *J. für die reine und angewandte Math.*, considered changes of variable $y = wv$, where w is a given function of x and v is the new variable that replaces y.

$$= \frac{au_0 e^{at}}{1 + bu_0 e^{at}} \qquad\qquad \text{Substitute } u = u_0 e^{at}.$$

$$= \frac{ae^{at}}{1/u_0 + be^{at}} \qquad\qquad \text{Divide by } u_0.$$

$$= \frac{ae^{at}}{(a - bP(0))/P(0) + be^{at}} \qquad \text{Use } u_0 = u(0) \text{ and } u = P/(a - bP).$$

$$= \frac{aP(0)e^{at}}{bP(0)e^{at} + a - bP(0)}. \qquad \text{Formula verified.}$$

The derivation using the substitution $u = P/(a - bP)$ requires only differential calculus. The substitution was found by afterthought, already knowing the solution; historically, integration methods have been applied.

The change of variables $(t, P) \to (t, u)$ given by the equation $u = P(a - bP)$ is used to justify the relation $u' = au$ as follows.

$$u' = \left(\frac{P}{a - bP}\right)' \qquad\qquad \text{It will be shown that } u' = au.$$

$$= \frac{P'(a - bP) - P(-bP')}{(a - bP)^2} \qquad \text{Quotient rule applied.}$$

$$= \frac{aP'}{(a - bP)^2} \qquad\qquad \text{Simplify the numerator.}$$

$$= \frac{a(a - bP)P}{(a - bP)^2} \qquad\qquad \text{Substitute } P' = (a - bP)P.$$

$$= au \qquad\qquad\qquad \text{Substitute } u = P/(a - bP).$$

This completes the motivation for the formula. To verify that it works in the differential equation is a separate issue, which is settled in the exercises.

Exercises 1.1

Growth-Decay Model. Solve the given initial value problem using the growth-decay formula; see page 3 and Example 1, page 7.

1. $y' = -3y$, $y(0) = 20$

2. $y' = 3y$, $y(0) = 1$

3. $3A' = A$, $A(0) = 1$

4. $4A' + A = 0$, $A(0) = 3$

5. $3P' - P = 0$, $P(0) = 10$

6. $4P' + 3P = 0$, $P(0) = 11$

7. $I' = 0.005I$, $I(t_0) = I_0$

8. $I' = -0.015I$, $I(t_0) = I_0$

9. $y' = \alpha y$, $y(t_0) = 1$

10. $y' = -\alpha y$, $y(t_0) = y_0$

Growth-decay Theory.

11. Graph without a computer $y = 10(2^x)$ on $-3 \le x \le 3$.

12. Graph without a computer $y = 10(2^{-x})$ on $-3 \le x \le 3$.

13. Find the doubling time for the growth model $P = 100e^{0.015t}$.

14. Find the doubling time for the growth model $P = 1000e^{0.0195t}$.

15. Find the elapsed time for the decay model $A = 1000e^{-0.11237t}$ until $|A(t)| < 0.00001$.

16. Find the elapsed time for the decay model $A = 5000e^{-0.01247t}$ until $|A(t)| < 0.00005$.

Newton Cooling Recipe. Solve the given cooling model. Follow Example 2 on page 7.

17. $u' = -10(u - 4)$, $u(0) = 5$

18. $y' = -5(y - 2)$, $y(0) = 10$

19. $u' = 1 + u$, $u(0) = 100$

20. $y' = -1 - 2y$, $y(0) = 4$

21. $u' = -10 + 4u$, $u(0) = 10$

22. $y' = 10 + 3y$, $y(0) = 1$

23. $2u' + 3 = 6u$, $u(0) = 8$

24. $4y' + y = 10$, $y(0) = 5$

25. $u' + 3(u + 1) = 0$, $u(0) = -2$

26. $u' + 5(u + 2) = 0$, $u(0) = -1$

27. $\alpha' = -2(\alpha - 3)$, $\alpha(0) = 10$

28. $\alpha' = -3(\alpha - 4)$, $\alpha(0) = 12$

Newton Cooling Model. The cooling model $u(t) = u_0 + A_0 e^{-ht}$ is applied; see page 4. Methods parallel those in the flask cooling example, page 9, and the baking example, page 10.

29. (**Ingot Cooling**) A metal ingot cools in the air at temperature 20C from 130C to 75C in one hour. Predict the cooling time to 23C.

30. (**Rod Cooling**) A plastic rod cools in a large vat of 12-degree Celsius water from 75C to 20C in 4 minutes. Predict the cooling time to 15C.

31. (**Murder Mystery**) A body discovered at 1:00 in the afternoon, March 1, 1929, had temperature 80F. Over the next hour the body's temperature dropped to 76F. Estimate the date and time of the murder.

32. (**Time of Death**) A dead person found in a 40F river had body temperature 70F. The coroner requested that the body be left in the river for 45 minutes, whereupon the body's temperature was 63F. Estimate the time of death, relative to the discovery of the body.

Verhulst Model. Solve the given Verhulst logistic equation using formula (8). Follow Example 3 on page 8.

33. $P' = P(2 - P)$, $P(0) = 1$

34. $P' = P(4 - P)$, $P(0) = 5$

35. $y' = y(y - 1)$, $y(0) = 2$

36. $y' = y(y - 2)$, $y(0) = 1$

37. $A' = A - 2A^2$, $A(0) = 3$

38. $A' = 2A - 5A^2$, $A(0) = 1$

39. $F' = 2F(3 - F)$, $F(0) = 2$

40. $F' = 3F(2 - F)$, $F(0) = 1$

Inverse Modeling. Given the model, find the differential equation and initial condition.

41. $A = A_0 e^{4t}$

42. $A = A_0 e^{-3t}$

43. $P = 1000e^{-0.115t}$

44. $P = 2000e^{-7t/5}$

45. $u = 1 + e^{-3t}$

46. $u = 10 - 2e^{-2t}$

47. $P = \dfrac{10}{10 - 8e^{-2t}}$

48. $P = \dfrac{5}{15 - 14e^{-t}}$

49. $P = \dfrac{1}{5 - 4e^{-t}}$

50. $P = \dfrac{2}{4 - 3e^{-t}}$

Populations. The following exercises use Malthusian population theory, page 5, and the Malthusian model $P(t) = P_0 e^{kt}$. Methods appear in Examples 4 and 5; see page 8.

51. (World Population) In June of 1993, the world population of $5,500,000,000$ people was increasing at a rate of $250,000$ people per day. Predict the date when the population reaches 10 billion.

52. (World Population) Suppose the world population at time $t = 0$ is 5 billion. How many years before that was the population one billion?

53. (Population Doubling) A population of rabbits increases by 10% per year. In how many years does the population double?

54. (Population Tripling) A population of bacteria increases by 15% per day. In how many days does the population triple?

55. (Population Growth) Trout in a river are increasing by 15% in 5 years. To what population size does 500 trout grow in 15 years?

56. (Population Growth) A region of 400 acres contains 1000 forest mushrooms per acre. The population is decreasing by 150 mushrooms per acre every 2 years. Find the population size for the 400-acre region in 15 years.

Verhulst Equation. Write out the solution to the given differential equation and, when it makes sense, report the carrying capacity

$$M = \lim_{t \to \infty} P(t).$$

57. $P' = (1 - P)P$

58. $P' = (2 - P)P$

59. $P' = 0.1(3 - 2P)P$

60. $P' = 0.1(4 - 3P)P$

61. $P' = 0.1(3 + 2P)P$

62. $P' = 0.1(4 + 3P)P$

63. $P' = 0.2(5 - 4P)P$

64. $P' = 0.2(6 - 5P)P$

65. $P' = 11P - 17P^2$

66. $P' = 51P - 13P^2$

Logistic Equation. The following exercises use the Verhulst logistic equation $P' = (a - bP)P$, page 6. Some methods appear on page 11.

67. (Protozoa) Experiments on the protozoa *Paramecium* determined growth rate $a = 2.309$ and carrying capacity $a/b = 375$ using initial population $P(0) = 5$. Establish the formula $P(t) = \dfrac{375}{1 + 74e^{-2.309t}}$.

68. (World Population) Demographers incorrectly projected the world population in the year 2000 as 6.5 billion (in 1970) and 5.9 billion (in 1976). Use $P(1965) = 3.358 \times 10^9$, $a = 0.029$ and carrying capacity $a/b = 1.0760668 \times 10^{10}$ to compute the logistic equation projection for year 2000.

69. (Harvesting) A fish population satisfying $P' = (a - bP)P$ is subjected to harvesting, the new model being $P' = (a - bP)P - H$. Assume $a = 0.04$, $a/b = 5000$ and $H = 1000$. Using algebra, rewrite it as $P' = b(\alpha - P)(P - \beta)$ in terms of the roots α, β of $ay - by^2 - H = 0$. Apply the change of variables $u = (\alpha - P)/(P - \beta)$ to solve it.

70. (Extinction) Let an endangered species satisfy $P' = bP^2 - aP$ for $a > 0$, $b > 0$. The term bP^2 represents births due to chance encounters of males and females, while the

term aP represents deaths. Use the change of variable $u = P/(bP - a)$ to solve it. Show from the answer that population sizes below a/b become extinct.

71. (Logistic Answer Check) Let $P = au/(1 + bu)$, $u = u_0 e^{at}$, $u_0 = P_0/(a - bP_0)$. Verify that $P(t)$ is a solution the differential equation $P' = (a - bP)P$ and $P(0) = P_0$.

72. (Logistic Equation) Let k, α, β be positive constants, $\alpha < \beta$. Solve $w' = k(\alpha - w)(\beta - w)$, $w(0) = w_0$ by the substitution $u = (\alpha - w)/(\beta - w)$, showing that $w = (\alpha - \beta u)/(1 - u)$, $u = u_0 e^{(\alpha - \beta)kt}$, $u_0 = (\alpha - w_0)/(\beta - w_0)$. This equation is a special case of the harvesting equation $P' = (a - bP)P + H$.

Growth-Decay Uniqueness Proof.

73. State precisely and give a calculus text reference for *Rolle's Theorem*, which says that a function vanishing at $x = a$ and $x = b$ must have slope zero at some point in $a < x < b$.

74. Apply Rolle's Theorem to prove that a differentiable function $v(x)$ with $v'(x) = 0$ on $a < x < b$ must be constant.

1.2 Exponential Application Library

The model differential equation $y' = ky$, and its variants via a change of variables, appears in various applications to biology, chemistry, finance, science and engineering. All the applications below use the exponential model $y = y_0 e^{kt}$.

Light Intensity	Chemical Reactions
Electric Circuits	Drug Elimination
Drug Dosage	Continuous Interest
Radioactive Decay	Radiocarbon Dating

Light Intensity

Physics defines the **lumen unit** to be the light flux through a solid unit angle from a point source of $1/621$ watts of yellow light.[2] The lumen is designed for measuring **brightness**, as perceived by the human eye. The **intensity** $E = \frac{F}{A}$ is the flux F per unit area A, with units Lux or Foot-candles (use $A = 1\text{m}^2$ or $A = 1\text{ft}^2$, respectively). At a radial distance r from a point source, in which case $A = 4\pi r^2$, the intensity is given by the **inverse square law**

$$E = \frac{F}{4\pi r^2}.$$

An **exposure meter**, which measures incident or reflected light intensity, consists of a body, a photocell and a readout in units of Lux or Foot-candles. Light falling on the photocell has energy, which is transferred by the photocell into electrical current and ultimately converted to the readout scale.

In classical physics experiments, a jeweler's bench is illuminated by a source of 8000 lumens. The experiment verifies the inverse square law, by reading an exposure meter at $1/2$, 1 and $3/2$ meters distance from the source.

As a variant on this experiment, consider a beaker of jeweler's cleaning fluid which is placed over the exposure meter photocell; see Figure 3. Successive meter readings with beaker depths of 0, 5, 10, 15 centimeters show that fluid **absorption** significantly affects the meter readings. Photons[3] striking the fluid convert into heat, which accounts for the rapid loss of intensity at depth in the fluid.

---

[2]Precisely, the wavelength of the light is 550-nm. The unit is equivalent to one **candela**, one of the seven basic SI units, which is the luminous intensity of one sixtieth of a square centimeter of pure platinum held at 1770C.

[3]A photon is the quantum of electromagnetic radiation, of energy $h\nu$, where ν is the radiation frequency and h is Planck's constant.
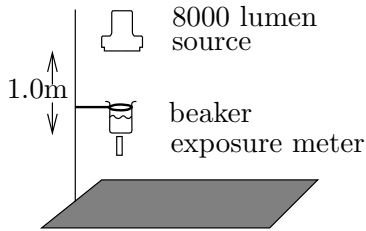
Figure 3. Jeweler's bench experiment.
The exposure meter measures light intensity at the beaker's base.

Empirical evidence from experiments suggests that light intensity $I(x)$ at a depth x in the fluid *changes at a rate proportional to itself*, that is,

(9) $$\frac{dI}{dx} = -kI.$$

If I_0 is the surface intensity at zero depth $(x = 0)$ and $I(x)$ is the intensity at depth x meters, then the theory of growth-decay equations applied to equation (9) gives the solution

(10) $$I(x) = I_0 e^{-kx}.$$

Equation (10) says that the intensity $I(x)$ at depth x is a percentage of the surface intensity $I(0) = I_0$, the percentage decreasing with depth x.

Electric Circuits

Classical physics analyzes the RC-circuit in Figure 4 and the LR-circuit in Figure 5. The physics background will be reviewed.
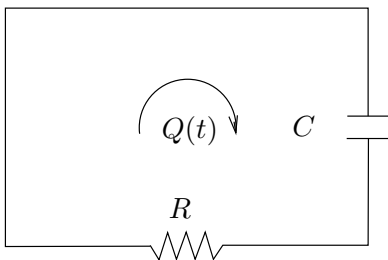

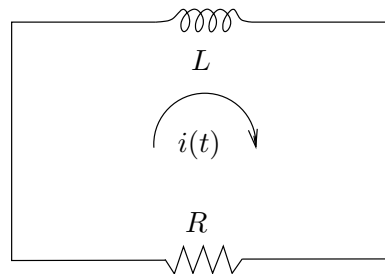**Figure 4. An RC-Circuit, no emf.**

Figure 5. An LR-Circuit, no emf.

First, the **charge** $Q(t)$ in coulombs and the **current** $I(t)$ in amperes are related by the rate formula $I(t) = Q'(t)$. We use prime notation $' = \frac{d}{dt}$. Secondly, there are some empirical laws that are used. There is **Kirchhoff's voltage law**:

> The algebraic sum of the voltage drops around a closed loop is zero.

Kirchhoff's **node law** is not used here, because only one loop appears in the examples.

There are the **voltage drop formulas** for an inductor of L henrys, a resistor of R ohms and a capacitor of C farads:

Faraday's law	$V_L = LI'$
Ohm's law	$V_R = RI$
Coulomb's law	$V_C = Q/C$

In Figure 4, Kirchhoff's law implies $V_R + V_C = 0$. The voltage drop formulas show that the charge $Q(t)$ satisfies $RQ'(t) + (1/C)Q(t) = 0$. Let $Q(0) = Q_0$. Growth-decay theory, page 3, gives $Q(t) = Q_0 e^{-t/(RC)}$.

In Figure 5, Kirchhoff's law implies that $V_L + V_R = 0$. By the voltage drop formulas, $LI'(t) + RI(t) = 0$. Let $I(0) = I_0$. Growth-decay theory gives $I(t) = I_0 e^{-Rt/L}$.

In summary:

RC-**Circuit**	$RQ' + (1/C)Q = 0, \quad Q(0) = Q_0,$
$Q = Q_0 e^{-t/(RC)}$	
LR-**Circuit**	$LI' + RI = 0, \quad I(0) = I_0.$
$I = I_0 e^{-Rt/L}$	

The ideas outlined here are illustrated in Examples 9 and 10, page 22.

Interest

The notion of **simple interest** is based upon the financial formula

$$A = (1 + r)^t A_0$$

where A_0 is the initial amount, A is the final amount, t is the number of years and r is the **annual interest rate** or **rate per annum** (5% means $r = 5/100$). The **compound interest** formula is

$$A = \left(1 + \frac{r}{n}\right)^{nt} A_0$$

where n is the number of times to compound interest per annum. Use $n = 4$ for **quarterly interest** and $n = 360$ for **daily interest**.

The topic of **continuous interest** has its origins in taking the limit as $n \to \infty$ in the compound interest formula. The answer to the limit problem is the **continuous interest formula**

$$A = A_0 e^{rt}$$

which by the growth-decay theory arises from the initial value problem

$$\begin{cases} A'(t) = rA(t), \\ A(0) = A_0. \end{cases}$$

Shown on page 27 are the details for taking the limit as $n \to \infty$ in the compound interest formula. In analogy with population theory, the following statement can be made about continuous interest.

> The amount accumulated by continuous interest increases at a rate proportional to itself.

Applied often in interest calculations is the **geometric sum formula**:

$$1 + r + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

The reader should pause to verify it for $n = 3$ by expanding the left side of the equivalent identity $(1 + r + r^2 + r^3)(r - 1) = r^4 - 1$.

Radioactive Decay

A constant fraction of the atoms present in a radioactive isotope will spontaneously decay into another isotope of the identical element or else into atoms of another element. Empirical evidence gives the following decay law:
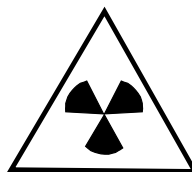
> A radioactive isotope decays at a rate proportional to the amount present.

In analogy with population models the differential equation for radioactive decay is

$$\frac{dA}{dt} = -kA(t),$$

where $k > 0$ is a physical constant called the **decay constant**, $A(t)$ is the number of atoms of radioactive isotope and t is measured in years.

Radiocarbon Dating. The decay constant $k \approx 0.0001245$ is known for carbon-14 (^{14}C). The model applies to measure the date that an organism died, assuming it metabolized atmospheric carbon-14.

The idea of radiocarbon dating is due to Willard S. Libby[4] in the late 1940s. The basis of the chemistry is that radioactive carbon-14, which has two more electrons than stable carbon-12, gives up an electron to become stable nitrogen-14. Replenishment of carbon-14 by cosmic rays keeps atmospheric carbon-14 at a nearly constant ratio with ordinary carbon-12 (this was Libby's assumption). After death, the radioactive decay of carbon-14 depletes the isotope in the organism. The percentage of depletion from atmospheric levels of carbon-14 gives a measurement that dates the organism.

---

[4]Libby received the Nobel Prize for Chemistry in 1960.

Definition 2 (Half-Life)
The **half-life** of a radioactive isotope is the time T required for half of the isotope to decay. In functional notation, it means $A(T) = A(0)/2$, where $A(t) = A(0)e^{kt}$ is the amount of isotope at time t.

For carbon-14, the half-life is 5568 years plus or minus 30 years, according to Libby (some texts and references give 5730 years). The decay constant $k \approx 0.0001245$ for carbon-14 arises by solving for $k = \ln(2)/5568$ in the equation $A(5568) = \frac{1}{2}A(0)$. Experts believe that carbon-14 dating methods tend to underestimate the age of a fossil.

Uranium-238 undergoes decay via alpha and beta radiation into various nuclides, the half-lives of which are shown in Table 1. The table illustrates the range of possible half-lives for a radioactive substance.

Table 1. Uranium-238 nuclides by alpha or beta radiation.

Nuclide	Half-Life
uranium-238	4,500,000,000 years
thorium-234	24.5 days
protactinium-234	1.14 minutes
uranium-234	233,000 years
thorium-230	83,000 years
radium-236	1,590 years
radon-222	3.825 days
polonium-218	3.05 minutes
lead-214	26.8 minutes
bismuth-214	19.7 minutes
polonium-214	0.00015 seconds
lead-210	22 years
bismuth-210	5 days
polonium-210	140 days
lead-206	stable

Tree Rings. Libby's work was based upon calculations from sequoia tree rings. Later investigations of 4000-year old trees showed that carbon ratios have been non-constant over past centuries.
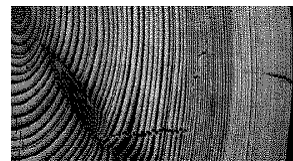
Libby's method is advertised to be useful for material 200 years to $40,000$ years old. Older material has been dated using the ratio of disintegration byproducts of potassium-40, specifically argon-40 to calcium-40.

An excellent reference for dating methods, plus applications and historical notes on the subject, is Chapter 1 of Braun [**?**].

Chemical Reactions

If the molecules of a substance decompose into smaller molecules, then an empirical law of **first-order reactions** says that the decomposition rate is proportional to the amount of substance present. In mathematical notation, this means

$$\frac{dA}{dt} = -hA(t)$$

where $A(t)$ is the amount of the substance present at time t and h is a physical constant called the **reaction constant**.

The **law of mass action** is used in chemical kinetics to describe **second-order reactions**. The law describes the amount $X(t)$ of chemical C produced by the combination of two chemicals A and B. A chemical derivation produces a rate equation

$$(11) \qquad X' = k(\alpha - X)(\beta - X), \quad X(0) = X_0,$$

where k, α and β are physical constants, $\alpha < \beta$; see Zill-Cullen [Z-C], Chapter 2. The substitution $u = (\alpha - X)/(\beta - X)$ is known to transform (11) into $u' = k(\alpha - \beta)u$ (see page 11 for the technique and the exercises in this section). Therefore,

$$(12) \qquad X(t) = \frac{\alpha - \beta u(t)}{1 - u(t)}, \quad u(t) = u_0 e^{(\alpha - \beta)kt}, \quad u_0 = \frac{\alpha - X_0}{\beta - X_0}.$$

Drug Elimination

Some drugs are eliminated from the bloodstream by an animal's body in a predictable fashion. The amount $D(t)$ in the bloodstream declines at a rate proportional to the amount already present. Modeling drug elimination exactly parallels radioactive decay, in that the translated mathematical model is

$$\frac{dD}{dt} = -hD(t),$$

where $h > 0$ is a physical constant, called the **elimination constant** of the drug.

Oral drugs must move through the digestive system and into the gut before reaching the bloodstream. The model $D'(t) = -hD(t)$ applies only after the drug has reached a stable concentration in the bloodstream and the body begins to eliminate the drug.

Examples

8 Example (Light intensity) Light intensity in a lake is decreased by 75% at depth one meter. At what depth is the intensity decreased by 95%?

Solution: The answer is 2.16 meters (7 feet, $1\frac{1}{16}$ inches). This depth will be justified by applying the light intensity model $I(x) = I_0 e^{-kx}$, where I_0 is the surface light intensity.

At one meter the intensity is $I(1) = I_0 e^{-k}$, but also it is given as $0.25 I_0$. The equation $e^{-k} = 0.25$ results, to determine $k = \ln 4 \approx 1.3862944$. To find the depth x when the intensity has decreased by 95%, solve $I(x) = 0.05 I_0$ for x. The value I_0 cancels from this equation, leaving $e^{-kx} = 1/20$. The usual logarithm methods give $x \approx 2.2$ meters, as follows:

$\ln e^{-kx} = \ln(1/20)$	Take the logarithm across $e^{-kx} = 1/20$.
$-kx = -\ln(20)$	Use $\ln e^u = u$ and $-\ln u = \ln(1/u)$.
$x = \dfrac{\ln(20)}{k}$	Divide by $-k$.
$= \dfrac{\ln(20)}{\ln(4)}$	Use $k = \ln(4)$.
≈ 2.16 meters.	Only 5% of the surface intensity remains at 2.16 meters.

9 Example (RC-Circuit) Solve the RC-circuit equation $RQ' + (1/C)Q = 0$ when $R = 2$, $C = 10^{-2}$ and the voltage drop across the capacitor at $t = 0$ is 1.5 volts.

Solution: The charge is $Q = 0.015 e^{-50t}$.

To justify this equation, start with the voltage drop formula $V_C = Q/C$, page 18. Then $1.5 = Q(0)/C$ implies $Q(0) = 0.015$. The differential equation is $Q' + 50Q = 0$. The solution from page 3 is $Q = Q(0)e^{-50t}$. Then the equation for the charge in coulombs is $Q(t) = 0.015 e^{-50t}$.

10 Example (LR-Circuit) Solve the LR-circuit equation $LI' + RI = 0$ when $R = 2$, $L = 0.1$ and the resistor voltage drop at $t = 0$ is 1.0 volts.

Solution: The solution is $I = 0.5 e^{-20t}$. To justify this equation, start with the voltage drop formula $V_R = RI$, page 18. Then $1.0 = RI(0)$ implies $I(0) = 0.5$. The differential equation is $I' + 20I = 0$; page 3 gives the solution $I = I(0)e^{-20t}$.

11 Example (Compound Interest) Compute the fixed monthly payment for a 5-year auto loan of $\$18,000$ at 9% per annum, using (a) daily interest and (b) continuous interest.

Solution: The payments are (a) \$373.94 and (b) \$373.95, which differ by one cent; details below.

Let $A_0 = 18000$ be the initial amount. It will be assumed that the first payment is due after 30 days and monthly thereafter. To simplify the calculation, a **day** is defined to be 1/360th of a year, regardless of the number of days in that year, and payments are applied every 30 days. Late fees apply if the payment is not received within the **grace period**, but it will be assumed here that all payments are made on time.

Part (a). The **daily interest rate** is $R = 0.09/360$ applied for 1800 periods (5 years). Between payments P, daily interest is applied to the balance $A(t)$ owed after t periods. The balance grows between payments and then decreases on the day of the payment. The problem is to *find P so that $A(1800) = 0$.*

Payments are subtracted every 30 periods making balance $A(30k)$. Let $B = (1 + R)^{30}$ and $A_k = A(30k)$. Then

$$
\begin{aligned}
A_k &= A(30k) && \text{Balance after } k \text{ months.}\\
&= A_0 B^k - P(1 + \cdots + B^{k-1}) && \text{For } k = 1, 2, 3, \ldots\\
&= A_0 B^k - P \frac{B^k - 1}{B - 1} && \text{Geometric sum formula applied, page 19.}
\end{aligned}
$$

$$
A_0 B^{60} = P \frac{B^{60} - 1}{B - 1} \qquad \text{Use } A(1800) = 0, \text{ which corresponds to } k = 60.
$$

$$
\begin{aligned}
P &= A_0 (B - 1) \frac{B^{60}}{B^{60} - 1} && \text{Solve for } P.\\
&= 373.93857 && \text{By calculator.}
\end{aligned}
$$

Part (b). The details are the same except for the method of applying interest. Let $s = 30(0.09)/360$, then

$$
\begin{aligned}
A_k &= A_0 e^{ks} - P e^{ks-s} \left(1 + e^{-s}\right. && \text{For } k = 1, 2, 3, \ldots, \text{ by examination of}\\
&\quad \left. + \cdots + e^{-ks+s}\right) && \text{cases } A(30) \text{ and } A(60).\\
&= A_0 e^{ks} - P e^{ks-s} \left(\frac{e^{-ks} - 1}{e^{-s} - 1}\right) && \text{Apply the geometric sum formula with common ratio } e^{-s}.
\end{aligned}
$$

$$
A_0 e^{60s} = P e^{60s-s} \frac{e^{-60s} - 1}{e^{-s} - 1} \qquad \text{Set } k = 60 \text{ and } A(1800) = 0 \text{ in the previous formula.}
$$

$$
\begin{aligned}
P &= A_0 \frac{-e^s + 1}{e^{-60s} - 1} && \text{Solve for } P.\\
&= 373.94604 && \text{By calculator.}
\end{aligned}
$$

12 Example (Effective Annual Yield) A bank advertises an effective annual yield of 5.73% for a certificate of deposit with continuous interest rate 5.5% per annum. Justify the rate.

Solution: The **effective annual yield** is the simple annual interest rate which gives the same account balance after one year. The issue is whether one year means 365 days or 360 days, since banks do business on a 360-day cycle.

Suppose first that one year means 365 days. The model used for a saving account is $A(t) = A_0 e^{rt}$ where $r = 0.055$ is the interest rate per annum. For one year, $A(1) = A_0 e^r$. Then $e^r = 1.0565406$, that is, the account has increased in one year by 5.65%. The *effective annual yield* is 0.0565 or 5.65%.

Suppose next that one year means 360 days. Then the bank pays 5.65% for only 360 days to produce a balance of $A_1 = A_0 e^r$. The extra 5 days make $5/360$ years, therefore the bank records a balance of $A_1 e^{5r/360}$ which is $A_0 e^{365r/360}$. The rate for 365 days is then 5.73%, by the calculation

$$
\frac{365}{360} 0.0565406 = 0.057325886.
$$

13 Example (Retirement Funds) An engineering firm offers a starting salary of 40 thousand per year, which is expected to increase 3% per year. Retirement contributions are 11% of salary, deposited monthly, growing at 6% continuous interest per annum. The company advertises a million dollars in retirement funds after 40 years. Justify the claim.

Solution: The salary is estimated to be $S(t) = 40000(1.03)^t$ after t years, because it starts with $S(0) = 40000$ and each year it takes a 3% increment. After 39 years of increases the salary has increased from $\$40,000$ to $\$126,681$.

Let A_n be the amount in the retirement account at the end of year n. Let $P_n = (40000(1.03)^n)(0.11)/12$ be the monthly salary for year $n+1$. The interest rates are $r = 0.06$ (annual) and $s = 0.06/12$ (monthly). For brevity, let $R = 1.03$.

During the first year, the retirement account accumulates 12 times for a total

$$A_1 = P_0 + \cdots + P_0 e^{11s}$$ Continuous interest at rate s on amount P_0 for 1 through 11 months.

$$= P_0 \frac{e^r - 1}{e^s - 1}$$ Geometric sum with common ratio e^s.

$$= 4523.3529.$$ Retirement balance after one year.

During the second and later years the retirement account accumulates by the rule

$$A_{n+1} = A_n e^r + P_n \\ + P_n e^s + \cdots + P_n e^{11s}$$ One year's accumulation at continuous rate r on amount A_n plus monthly accumulations on retirement contributions P_n.

$$= A_n e^r + P_n \frac{e^r - 1}{e^s - 1}$$ Apply the geometric sum formula with common ratio e^s.

$$= A_n e^r + R^n P_0 \frac{e^r - 1}{e^s - 1}$$ Use $P_n = P_0 R^n$.

$$= A_n e^r + R^n A_1.$$ Apply $A_1 = P_0 \frac{e^r - 1}{e^s - 1}$.

After examining cases $n = 1, 2, 3$, the recursion is solved to give

$$A_n = A_1 \sum_{k=0}^{n-1} e^{kr} R^{n-1-k}.$$

To establish this formula, induction is applied:

$$A_{n+1} = A_n e^r + R^n A_1$$ Derived above.

$$= A_1 e^r \sum_{k=0}^{n-1} e^{kr} R^{n-1-k} + R^n A_1$$ Apply the induction hypothesis.

$$= A_1 \sum_{k=0}^{n} e^{kr} R^{n-k}$$ Rewrite the sum indices.

$$= A_1 R^n \frac{(e^r/R)^{n+1} - 1}{e^r/R - 1}.$$ Use the geometric sum formula with common ratio e^r/R.

The advertised retirement fund after 40 years should be the amount A_{40}, which is obtained by setting $n = 39$ in the last equality. Then $A_{40} = 1102706.60$.

14 Example (Half-life of Radium) A radium sample loses $1/2$ percent due to disintegration in 12 years. Verify the half-life of the sample is about $1,660$ years.

Solution: The decay model $A(t) = A_0 e^{-kt}$ applies. The given information $A(12) = 0.995A(0)$ reduces to the exponential equation $e^{-12k} = 0.995$ with solution $k = \ln(1000/995)/12$. The half-life T satisfies $A(T) = \frac{1}{2}A(0)$, which reduces to $e^{-kT} = 1/2$. Since k is known, the value T can be found as $T = \ln(2)/k \approx 1659.3909$ years.

15 Example (Radium Disintegration) The disintegration reaction

$$_{88}R^{226} \longrightarrow \, _{88}R^{224}$$
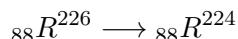
of radium-226 into radon has a half-life of 1700 years. Compute the decay constant k in the decay model $A' = -kA$.

Solution: The half-life equation is $A(1700) = \frac{1}{2}A(0)$. Since $A(t) = A_0 e^{-kt}$, the equation reduces to $e^{-1700k} = 1/2$. The latter is solved for k by logarithm methods (see page 8), giving $k = \ln(2)/1700 = 0.00040773364$.

16 Example (Radiocarbon Dating) The ratio of carbon-14 to carbon-12 in a dinosaur fossil is 6.34 percent of the current atmospheric ratio. Verify the dinosaur's death was about $22,160$ years ago.

Solution: The method due to Willard Libby will be applied, using his assumption that the ratio of carbon-14 to carbon-12 in living animals is equal to the atmospheric ratio. Then carbon-14 depletion in the fossil satisfies the decay law $A(t) = A_0 e^{-kt}$ for some parameter values k and A_0.

Assume the half-life of carbon-14 is 5568 years. Then $A(5568) = \frac{1}{2}A(0)$ (see page 20). This equation reduces to $A_0 e^{-5568k} = \frac{1}{2}A_0 e^0$ or $k = \ln(2)/5568$. In short, k is known but A_0 is unknown. It is not necessary to determine A_0 in order to do the verification.

At the time t_0 in the past when the organism died, the amount A_1 of carbon-14 began to decay, reaching the value $6.34A_1/100$ at time $t = 0$ (the present). Therefore, $A_0 = 0.0634A_1$ and $A(t_0) = A_1$. Taking this last equation as the starting point, the final calculation proceeds as follows.

$A_1 = A(t_0)$	The amount of carbon-14 at death is A_1, $-t_0$ years ago.
$\quad = A_0 e^{-kt_0}$	Apply the decay model $A = A_0 e^{-kt}$ at $t = t_0$.
$\quad = 0.0634A_1 e^{-kt_0}$	Use $A_0 = 6.34A_1/100$.

The value A_1 cancels to give the new relation $1 = 0.0634e^{-kt_0}$. The value $k = \ln(2)/5568$ gives an exponential equation to solve for t_0:

$e^{kt_0} = 0.0634$	Multiply by e^{kt_0} to isolate the exponential.
$\ln e^{kt_0} = \ln(0.0634)$	Take the logarithm of both sides.
$t_0 = \dfrac{1}{k} \ln(0.0634)$	Apply $\ln e^u = u$ and divide by k.
$= \dfrac{5568}{\ln 2} \ln(0.0634)$	Substitute $k = \ln(2)/5568$.
$= -22157.151$ years.	By calculator. The fossil's age is the negative.

17 Example (Percentage of an Isotope) A radioactive isotope disintegrates by 5% in ten years. By what percentage does it disintegrate in one hundred years?

Solution: The answer is not 50%, as is widely reported by lay persons. The correct answer is 40.13%. It remains to justify this non-intuitive answer.

The model for decay is $A(t) = A_0 e^{-kt}$. The decay constant k is known because of the information *... disintegrates by 5% in ten years*. Translation to equations produces $A(10) = 0.95 A_0$, which reduces to $e^{-10k} = 0.95$. Solving with logarithms gives $k = 0.1 \ln(100/95) \approx 0.0051293294$.

After one hundred years, the isotope present is $A(100)$, and the percentage is $100 \frac{A(100)}{A(0)}$. The common factor A_0 cancels to give the percentage $100 e^{-100k} \approx 59.87$. The reduction is 40.13%.

To reconcile the lay person's answer, observe that the amounts present after one, two and three years are $0.95 A_0$, $(0.95)^2 A_0$, $(0.95)^3 A_0$. The lay person should have guessed 100 times $1 - (0.95)^{10}$, which is 40.126306. The common error is to simply multiply 5% by the ten periods of ten years each. By this erroneous reasoning, the isotope would be depleted in two hundred years, whereas the decay model says that about 36% of the isotope remains!

18 Example (Chemical Reaction) The manufacture of t-butyl alcohol from t-butyl chloride is made by the chemical reaction

$$(CH_3)_3 CCL + NaOH \longrightarrow (CH_3)_3 COH + NaCL.$$
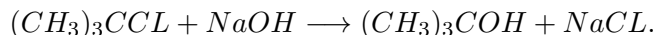
Model the production of t-butyl alcohol, when $N\%$ of the chloride remains after t_0 minutes.

Solution: It will be justified that the model for alcohol production is $A(t) = C_0(1 - e^{-kt})$ where $k = \ln(100/N)/t_0$, C_0 is the initial amount of chloride and t is in minutes.

According to the theory of first-order reactions, the model for chloride depletion is $C(t) = C_0 e^{-kt}$ where C_0 is the initial amount of chloride and k is the reaction constant. The alcohol production is $A(t) = C_0 - C(t)$ or $A(t) = C_0(1 - e^{-kt})$. The reaction constant k is found from the initial data $C(t_0) = \frac{N}{100} C_0$, which results in the exponential equation $e^{-kt_0} = N/100$. Solving the exponential equation gives $k = \ln(100/N)/t_0$.

19 Example (Drug Dosage) A veterinarian applies general anesthesia to animals by injection of a drug into the bloodstream. Predict the drug dosage to anesthetize a 25-pound animal for thirty minutes, given:

1. The drug requires an injection of 20 milligrams per pound of body weight in order to work.

2. The drug eliminates from the bloodstream at a rate proportional to the amount present, with a half-life of 5 hours.

Solution: The answer is about 536 milligrams of the drug. This amount will be justified using exponential modeling.

The drug model is $D(t) = D_0 e^{-ht}$, where D_0 is the initial dosage and h is the elimination constant. The half-life information $D(5) = \frac{1}{2}D_0$ determines $h = \ln(2)/5$. Depletion of the drug in the bloodstream means the drug levels are always decreasing, so it is enough to require that the level at 30 minutes exceeds 20 times the body weight in pounds, that is, $D(1/2) > (20)(25)$. The critical value of the initial dosage D_0 then occurs when $D(1/2) = 500$ or $D_0 = 500e^{h/2} = 500e^{0.1\ln(2)}$, which by calculator is approximately 535.88673 milligrams.

Drugs like sodium pentobarbital behave somewhat like this example, although injection in a single dose is unusual. An intravenous drip can sustain the blood levels of the drug, keeping the level closer to the target 500 milligrams.

Details and Proofs

Verification of Continuous Interest by Limiting. Derived here is the continuous interest formula by limiting as $n \to \infty$ in the compound interest formula.

$$\left(1 + \frac{r}{n}\right)^{nt} = B^{nt} \qquad \text{In the exponential rule } B^x = e^{x\ln B}, \text{ the base is } B = 1 + r/n.$$

$$= e^{nt\ln B} \qquad \text{Use } B^x = e^{x\ln B} \text{ with } x = nt.$$

$$= e^{\frac{r\ln(1+u)}{u}t} \qquad \text{Substitute } u = r/n. \text{ Then } u \to 0 \text{ as } n \to \infty.$$

$$\approx e^{rt} \qquad \text{Because } \ln(1+u)/u \approx 1 \text{ as } u \to 0, \text{ by L'Hospital's rule.}$$

Exercises 1.2

Light Intensity. The following exercises apply the theory of light intensity on page 16, using the model $I(t) = I_0 e^{-kx}$ with x in meters. Methods parallel Example 8 on page 21.

1. The light intensity is $I(t) = I_0 e^{-1.4x}$ in a certain swimming pool. At what depth does the light intensity fall off by 50%?

2. The light intensity in a swimming

pool falls off by 50% at a depth of 2.5 meters. Find the depletion constant k in the exponential model.

3. Plastic film is used to cover window glass, which reduces the interior light intensity by 10%. By what percentage is the intensity reduced, if two layers are used?

4. Double-thickness colored window glass is supposed to reduce the interior light intensity by 20%. What is the reduction for single-thickness colored glass?

RC-Electric Circuits. In the exercises below, solve for $Q(t)$ when $Q_0 = 10$ and graph $Q(t)$ on $0 \le t \le 5$.

5. $R = 1$, $C = 0.01$.

6. $R = 0.05$, $C = 0.001$.

7. $R = 0.05$, $C = 0.01$.

8. $R = 5$, $C = 0.1$.

9. $R = 2$, $C = 0.01$.

10. $R = 4$, $C = 0.15$.

11. $R = 4$, $C = 0.02$.

12. $R = 50$, $C = 0.001$.

LR-Electric Circuits. In the exercises below, solve for $I(t)$ when $I_0 = 5$ and graph $I(t)$ on $0 \le t \le 5$.

13. $L = 1$, $R = 0.5$.

14. $L = 0.1$, $R = 0.5$.

15. $L = 0.1$, $R = 0.05$.

16. $L = 0.01$, $R = 0.05$.

17. $L = 0.2$, $R = 0.01$.

18. $L = 0.03$, $R = 0.01$.

19. $L = 0.05$, $R = 0.005$.

20. $L = 0.04$, $R = 0.005$.

Interest and Continuous Interest. Financial formulas which appear on page 18 are applied below, following the ideas in Examples 11, 12 and 13, pages 22–24.

21. (**Total Interest**) Compute the total daily interest and also the total continuous interest for a 10-year loan of $5,000$ dollars at 5% per annum.

22. (**Total Interest**) Compute the total daily interest and also the total continuous interest for a 15-year loan of $7,000$ dollars at $5\frac{1}{4}\%$ per annum.

23. (**Monthly Payment**) Find the monthly payment for a 3-year loan of $8,000$ dollars at 7% per annum compounded continuously.

24. (**Monthly Payment**) Find the monthly payment for a 4-year loan of $7,000$ dollars at $6\frac{1}{3}\%$ per annum compounded continuously.

25. (**Effective Yield**) Determine the effective annual yield for a certificate of deposit at $7\frac{1}{4}\%$ interest per annum, compounded continuously.

26. (**Effective Yield**) Determine the effective annual yield for a certificate of deposit at $5\frac{3}{4}\%$ interest per annum, compounded continuously.

27. (**Retirement Funds**) Assume a starting salary of $35,000$ dollars per year, which is expected to increase 3% per year. Retirement contributions are $10\frac{1}{2}\%$ of salary, deposited monthly, growing at $5\frac{1}{2}\%$ continuous interest per annum. Find the retirement amount after 30 years.

28. (**Retirement Funds**) Assume a starting salary of $45,000$ dollars per year, which is expected to increase 3% per year. Retirement contributions are $9\frac{1}{2}\%$ of

salary, deposited monthly, growing at $6\frac{1}{4}\%$ continuous interest per annum. Find the retirement amount after 30 years.

29. **(Actual Cost)** A van is purchased for $18,000$ dollars with no money down. Monthly payments are spread over 8 years at $12\frac{1}{2}\%$ interest per annum, compounded continuously. What is the actual cost of the van?

30. **(Actual Cost)** Furniture is purchased for $15,000$ dollars with no money down. Monthly payments are spread over 5 years at $11\frac{1}{8}\%$ interest per annum, compounded continuously. What is the actual cost of the furniture?

Radioactive Decay. Assume the decay model $A' = -kA$ from page 19. Below, $A(T) = 0.5A(0)$ defines the *half-life* T. Methods parallel Examples 14–17 on pages 25–26.

31. **31.**(Half-Life) Determine the half-life of a radium sample which decays by 5.5% in 13 years.

32. **(Half-Life)** Determine the half-life of a radium sample which decays by 4.5% in 10 years.

33. **(Half-Life)** Assume a radioactive isotope has half-life 1800 years. Determine the percentage decayed after 150 years.

34. **(Half-Life)** Assume a radioactive isotope has half-life 1650 years. Determine the percentage decayed after 99 years.

35. **(Disintegration Constant)** Determine the constant k in the model $A' = -kA$ for radioactive material that disintegrates by 5.5% in 13 years.

36. **(Disintegration Constant)** Determine the constant k in the model $A' = -kA$ for radioactive material that disintegrates by 4.5% in 10 years.

37. **(Radiocarbon Dating)** A fossil found near the town of Dinosaur, Utah contains carbon-14 at a ratio of 6.21% to the atmospheric value. Determine its approximate age according to Libby's method.

38. **(Radiocarbon Dating)** A fossil found in Colorado contains carbon-14 at a ratio of 5.73% to the atmospheric value. Determine its approximate age according to Libby's method.

39. **(Radiocarbon Dating)** In 1950, the Lascaux Cave in France contained charcoal with 14.52% of the carbon-14 present in living wood samples nearby. Estimate by Libby's method the age of the charcoal sample.

40. **(Radiocarbon Dating)** At an excavation in 1960, charcoal from building material had 61% of the carbon-14 present in living wood nearby. Estimate the age of the building.

41. **(Percentage of an Isotope)** A radioactive isotope disintegrates by 5% in 12 years. By what percentage is it reduced in 99 years?

42. **(Percentage of an Isotope)** A radioactive isotope disintegrates by 6.5% in $1,000$ years. By what percentage is it reduced in $5,000$ years?

Chemical Reactions. Assume below the model $A' = kA$ for a first-order reaction. See page 21 and Example 18, page 26.

43. **(First-Order $A + B \longrightarrow C$)** A first order reaction produces product C from chemical A and catalyst B. Model the production of C, given $N\%$ of A remains after t_0 minutes.

44. (First-Order $A + B \longrightarrow C$) A first order reaction produces product C from chemical A and catalyst B. Model the production of C, given $M\%$ of A is depleted after t_0 minutes.

45. (Law of Mass-Action) Consider a second-order chemical reaction $X(t)$ with $k = 0.14$, $\alpha = 1$, $\beta = 1.75$, $X(0) = 0$. Find an explicit formula for $X(t)$ and graph it on $t = 0$ to $t = 2$.

46. (Law of Mass-Action) Consider a second-order chemical reaction $X(t)$ with $k = 0.015$, $\alpha = 1$, $\beta = 1.35$, $X(0) = 0$. Find an explicit formula for $X(t)$ and graph it on $t = 0$ to $t = 10$.

47. (Mass-Action Derivation) Let k, α, β be positive constants, $\alpha < \beta$. Solve $X' = k(\alpha - X)(\beta - X)$, $X(0) = X_0$ by the substitution $u = (\alpha - X)/(\beta - X)$, showing that $X = (\alpha - \beta u)/(1 - u)$, $u = u_0 e^{(\alpha - \beta)kt}$, $u_0 = (\alpha - X_0)/(\beta - X_0)$.

48. (Mass-Action Derivation) Let k, α, β be positive constants, $\alpha < \beta$. Define $X = (\alpha - \beta u)/(1 - u)$, where $u = u_0 e^{(\alpha - \beta)kt}$ and $u_0 = (\alpha - X_0)/(\beta - X_0)$. Verify by calculus computation that (1) $X' = k(\alpha - X)(\beta - X)$ and (2) $X(0) = X_0$.

Drug Dosage. Employ the drug dosage model $D(t) = D_0 e^{-ht}$ given on page 21. Let h be determined by a half-life of three hours. Apply the techniques of Example 19, page 27.

49. (Injection Dosage) Bloodstream injection of a drug into an animal requires a minimum of 20 milligrams per pound of body weight. Predict the dosage for a 12-pound animal which will maintain a drug level 3% higher than the minimum for two hours.

50. (Injection Dosage) Bloodstream injection of an antihistamine into an animal requires a minimum of 4 milligrams per pound of body weight. Predict the dosage for a 40-pound animal which will maintain an antihistamine level 5% higher than the minimum for twelve hours.

51. (Oral Dosage) An oral drug with first dose 250 milligrams is absorbed into the bloodstream after 45 minutes. Predict the number of hours after the first dose at which to take a second dose, in order to maintain a blood level of at least 180 milligrams for three hours.

52. (Oral Dosage) An oral drug with first dose 250 milligrams is absorbed into the bloodstream after 45 minutes. Determine three (small) dosage amounts, and their administration time, which keep the blood level above 180 milligrams but below 280 milligrams over three hours.

1.3 Differential Equations

Defined here are first order differential equations. The nature of a solution is studied through possible representations as explicit or implicit equations, numeric tables and graphical visualization.

First Order Differential Equation

The equation

(1) $$y'(x) = f(x, y(x))$$

is called a **first order differential equation**. The function $f(x, y)$ is defined in a region D of the xy-plane. In most physical applications f is continuous in D or else it has simple discontinuities, such as those caused by switches.

Cited below are some striking examples of first order differential equations in science and engineering.

$y' = F(x)$ — $\int_{x_0}^{x} F(t)dt$ represent solutions to differential equations, because of the fundamental theorem of calculus, page 873.

$u' = -k(u - u_1)$ — **Cooling** of a body with temperature $u(t)$ in a medium of temperature u_1 obeys Newton's law of cooling. Symbol k is the **cooling constant**.

$Q' = k(T^4 - T_0^4)$ — **Stefan's radiation law** models the heat lost by a body of temperature T in a medium of temperature T_0 due to thermal radiation.

$\dot{y} = -h\sqrt{y}$ — **Tank draining** obeys **Torricelli's law**, where h is a constant and y is the fluid depth in the tank at time t.

$\dfrac{dP}{dt} = kP$ — **Population dynamics** may assume Malthus's reproduction law: *the population changes at a rate proportional to the present population P.*

$\dot{v} = F/m$ — **Free fall** velocity v of a mass m accelerating due to constant gravitational force F obeys Newton's second law $F = ma$, where a is the acceleration.

$y' = k(a^2 - x^2)$ — **Boat trajectories** for a river crossing, with the fastest current in the center, can be modelled by the distance x from the center and the distance $y(x)$ downstream.

Solutions and Answers

A quadratic equation $ax^2 + bx + c = 0$ has *numerical answer* $x = -b/(2a) \pm \sqrt{b^2 - 4ac}/(2a)$. Differential equations have answers that are graphs, represented by *functions* $y(x)$. Sadly, it is generally impossible to write down a formula for the **answer** $y(x)$ to a given differential equation

$$\frac{d}{dx}y(x) = f(x, y(x)).$$

Applied Models. Scientists and engineers who do modeling are not much interested in *solving a differential equation*. They use differential equations to express or define a mathematical model. Initially, during modeling stages, theoretical existence of the function is sufficient. Later on, analytical and numerical methods must be applied to actually find the function. In summary:

> Differential equations are used in application modeling to **define** or **express** a function of the physical parameters.

Answers. An answer to a differential equation problem is given in various forms, suited to the intended application. The most common forms are **tables**, **equations** and **graphs**. Answers are related to the notion of a **solution**, which is a precise mathematical term, defined below.

Definition 3 (Solution)
Let f be defined for $a < x < b$ and $c < y < d$. A **solution** y to the differential equation $y' = f(x, y)$ on the interval (a, b) is a function $y(x)$ defined for $a < x < b$ such that

> (1) The left side $y'(x)$ of the differential equation and the right side $f(x, y(x))$ are defined for each $a < x < b$.

> (2) Substitution of $y(x)$ in each side gives symbolically equal expressions for each value of x in the domain $a < x < b$.

Often solution formulas contain physical constants represented as symbols. In such cases the definition is modified to say *each side gives symbolically equal expressions for all symbols.*

The definition can be restated for half-open intervals, closed intervals and intervals in which one or both endpoints are infinite. If $f(x, y)$ contains discontinuous switches, then the definition of solution is relaxed, possibly excluding points of discontinuity.

Definition 4 (Equilibrium Solution)
An **equilibrium solution** y to the differential equation $y' = f(x, y)$ is
a constant solution $y(x) = k$ of the differential equation.

To illustrate, $y' = y(1 - y)$ has two equilibrium solutions $y = 0$ and
$y = 1$. They are found by formal substitution of $y = k$ into $y' = y(1 - y)$
and then solving for k in the formal equation $0 = k(1 - k)$. For more
information on equilibrium solutions, see page 84, Example 7 on page 87
and Example 9 on page 88.

The equation $y' = x(1 - y)$ has equilibrium solution $y = 1$. The equation
$x = 0$, often reported by rookie differential equations students, is not an
equilibrium solution. The formal equation $0 = x(1 - k)$ is solved for k
with symbol x allowed to assume all possible values. Then $x \neq 0$ forces
$k = 1$. We must report equilibrium solution $y = 1$.

Definition 5 (Initial Value Problem)
The **initial value problem** for a first order equation $y' = f(x, y)$ on
$a < x < b$ is the problem of finding a solution y on $a < x < b$ which in
addition satisfies an **initial condition** of the form $y = y_0$ at $x = x_0$. An
initial condition may be given in compact notation $y(x_0) = y_0$.

To make sense of the initial condition, $f(x, y)$ must have (x_0, y_0) in its
domain of definition, that is, $a < x_0 < b$ and $c < y_0 < d$. Similar
statements apply to more general domains.

Uniqueness. In typical applications, just one solution is isolated by
the initial condition. Having just one solution is not obvious on physical
grounds; see Example 20. **Non-uniqueness** allows modeling an answer
like $y = 1 + x^3$ through an initial value problem, while a numerical
procedure computes a different answer like $y = 1$. **Uniqueness** forces
the modeler and the solver to find the same answer. The jobs of scientists and engineers include keeping computers from producing nonsense
numbers and incorrect graphs. It is possible for bad modeling, which
allows non-uniqueness, to cause bad results to come off the computer.
In summary:

> Numerical answers and computer graphs obtained from the
> differential equation $y' = f(x, y)$ are **nonsense** unless the
> model has a unique solution.

Explicit and Implicit Equations

Equations that represent answers to first order differential equations are
either **implicit** or **explicit**. An equation with y isolated on the left side

and right side independent of y is called *explicit*. Otherwise, the equation is called *implicit*. Some examples:

$y = \sin x + e^{-x}$ Usual equations treated in differential calculus are *explicit* equations.

$y = f(x)$ Equations given in abstract functional notation are *explicit* equations.

$y = 1 + \pi$ Constant equations are *explicit* equations.

$2y = 1$ An *implicit* equation (y not isolated left). Can be converted to explicit equation $y = 1/2$.

$x + y = 1$ As written, y is not isolated on the left, so it is an *implicit* equation. It can be converted to the *explicit* equation $y = 1 - x$.

$x^2 + y^2 = 1$ The equation of a circle is an *implicit* equation.

$f(x, y) = c$ Abstract level curve equations are assumed to be in *implicit* form. To convert to *explicit* form, solve for y in terms of x.

$x + y^2 = 1$ As written, y is not isolated on the left, so it is an *implicit* equation. It converts to two *explicit* equations $y = \sqrt{1 - x}$ and $y = -\sqrt{1 - x}$.

Definition 6 (Explicit Equation)
An xy-equation is **explicit** if exactly y appears on the left, followed by an equal sign, followed by an expression independent of y. In functional notation, the equation must have the form $y = f(x)$.

Any equation that is not explicit is called **implicit**.

Equations $2y = x$, $-y = 1 + x$ and $xy = 1$ are implicit, but they can be converted by algebra into the explicit equations $y = x/2$, $y = -1 - x$, $y = 1/x$.

Numeric Tables

A **numeric table** is a list of x, y values. Tables are finite lists. Typical numeric tables appear in Examples 22 and 23 on page 36.

A numeric table for a differential equation $y' = f(x, y)$ can be generated by a **numerical method**. Normally, the x-values are equally spaced on some interval. A specific numerical method is applied to find each of the y-values. The most elementary numerical methods are Euler's method, Heun's method and the Runge-Kutta method.

A numeric table in current scientific literature may assume that x or y is a *vector variable*. The effect is to allow numeric tables with multiple columns.

Graphics

Graphs of solutions to differential equations $y' = f(x, y)$ can be generated by hand from numeric data. The most popular method for hand-graphing is the **connect-the-dots method**. This method constructs a graph as straight-line connections of the data points. See Example 24, page 37.

Curve library methods and computer methods for graphing equations and numerical data sets are considered elsewhere; see page 880. These methods apply especially if the curve is given by an equation, explicit or implicit.

Examples

20 Example (IVP with Two Solutions) Display an answer check for the initial value problem (IVP) on interval $x \geq 0$, showing that it has two solutions: (1) $y(x) = x^2/4$ and (2) constant solution $y(x) = 0$.

$$y' = \sqrt{|y|}, \quad y(0) = 0,$$

Solution: The example is important, because modern computer algebra systems allow numeric methods to be blindly applied to examples like this one. No error messages are emitted by such computer programs. Failures, often blamed on computers, can be the result of unexpected modeling intricacies.

The example is curiously close to the tank-draining equation $y' = -h\sqrt{y}$ based upon *Torricelli's law*, page 31. Arguments that an equation *physically has a unique solution* are unheard by computer programs: the programs are not smarter than the humans who use them.

The verification involves two steps:

(a) The differential equation $y' = \sqrt{|y|}$ has y as a solution.

(b) The initial condition $y(0) = 0$ ($y = 0$ at $x = 0$) is satisfied.

Answer Check for Solution $y(x) = x^2/4$.
In both steps (a) and (b), the verification amounts to expanding the left hand side (LHS) and right hand side (RHS) of the equalities, then a check is made for equality of the LHS and RHS, for all symbols. The details for $y = x^2/4$ are as follows.

LHS $= y'$	The left side of $y' = \sqrt{|y|}$ is y'.
$= (x^2/4)'$	The solution being tested is $y = x^2/4$.
$= \dfrac{x}{2}$,	and
RHS $= \sqrt{|y|}$	The right side of $y' = \sqrt{|y|}$.
$= \sqrt{|x^2/4|}$	Because $y = x^2/4$.
$= x/2$	Because $x \geq 0$.

Therefore, LHS = RHS, and step (a) is finished. To complete step (b), proceed similarly with the initial condition $y(0) = 0$:

LHS $= y(0)$ The left side of the initial condition $y(0) = 0$.

$= \left. \dfrac{x^2}{4} \right|_{x=0}$ The solution being tested is $y = x^2/4$.

$= 0$

$= $ RHS   The right side of $y(0) = 0$.

Answer Check for Solution $y(x) = 0$.
The details for the constant solution $y(x) = 0$ are similar. As a mental exercise, repeat the steps above with $x^2/4$ replaced by 0, to verify steps (a) and (b) for the constant solution $y(x) = 0$.

21 Example (Implicit and Explicit Equations) Classify $1 + e^y = x^2$ as implicit or explicit. If implicit, then find an explicit representation for y in terms of x.

Solution: The equation is classified as *implicit*, because y is not isolated on the left side. Conversion to explicit form uses college algebra, as follows.

$1 + e^y = x^2$ Given equation. Solving for y.

$e^y = x^2 - 1$ Isolate y-terms on the left.

$\ln e^y = \ln |x^2 - 1|$ Take the logarithm of both sides.

$y = \ln |x^2 - 1|$ Simplify the left side. Identity $\ln e^u = u$ applied.

22 Example (Verify a Numerical Table) Verify Table 2 using the explicit equation $y = 1 - x + 2x^2$.

x	0.0	0.1	0.2	0.3	0.4
y	1.0	0.92	0.88	0.88	0.92

Table 2. Numerical data for an explicit equation.

Solution: Each column is verified in an identical way. For example, column 2 is checked by substitution of $x = 0.1$ into $y = 1 - x + 2x^2$ to obtain $y = 1 - 0.1 + 2(0.1)^2 = 0.92$.

23 Example (Verify an Approximation Table) Verify Table 3 using the approximation formula $y(x + 0.1) \approx y(x) + 0.1(x + y(x))$.

x	y
0.0	1.0
0.1	1.1
0.2	1.22
0.3	1.362
0.4	1.5282

Table 3. Numerical data for an approximation formula.

Solution: The formula is applied as a **recursion**, which is a set of formulas which generate from a *given table pair* x, y the *next table pair* X, Y via the relations

$$X = x + 0.1, \quad Y = y + 0.1(x + y).$$

Important in the mathematical translation is the elimination of the approximation symbol ($\approx$) and the use of equal signs ($=$) in the final relations.

Each row is verified in an identical way. For example, row 3 is checked by substitution of data from the previous row. Items $x = 0.1$ and $y = 1.1$ from row 2 are substituted into $X = x + 0.1$ and $Y = y + 0.1(x + y)$ to obtain $X = 0.2$ and $Y = 1.22$. The approximations 0.2, 1.22 are then copied to row 3 of the table.

24 Example (Hand Graphing of Numeric Data) Graph on engineering paper the piecewise-defined function $y(x)$ using six data points from $x = 0$ to $x = 1/2$ in steps of 0.1.

$$y(x) = \begin{cases} 1.1x + 1.10 & 0.0 \le x \le 0.1, \\ -1.6x + 1.37 & 0.1 < x \le 0.2, \\ 1.5x + 0.75 & 0.2 < x \le 0.3, \\ 0.1x + 0.90 & 0.3 < x \le 0.4, \\ -0.2x + 2.10 & 0.4 < x \le 0.5. \end{cases}$$

Solution: The xy-data points for $y(x)$ are

$$(0.0, 1.10), \quad (0.1, 1.21), \quad (0.2, 1.05),$$
$$(0.3, 1.20), \quad (0.4, 1.30), \quad (0.5, 1.10).$$

Engineering paper divisions are set for this example at 0.1 horizontal and 0.1 vertical. The origin will be $x = 0.0$, $y = 1.0$. The first step is to plot the points as *dots*. The second step connects the dots with straight lines, just as in children's *connect-the-dot* puzzles. The graphic appears in Figure 6. The figure is correct between data points because $y(x)$ is piecewise linear. Generally, the connect-the-dot method makes errors between data points.
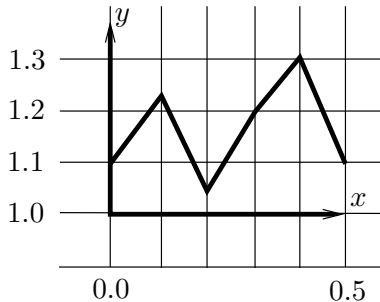

Figure 6. Engineering paper graphic of numeric data.

Exercises 1.3

Solution Verification. Given the differential equation, initial condition and proposed solution y, verify that y is a solution. Don't try to *solve* the

equation!

1. $\dfrac{dy}{dx} = y$, $y(0) = 2$, $y = 2e^x$

2. $y' = 2y$, $y(0) = 1$, $y = e^{2x}$

3. $y' = y^2$, $y(0) = 1$, $y = (1-x)^{-1}$

4. $\dfrac{dy}{dx} = y^3$, $y(0) = 1$,
$y = (1-2x)^{-1/2}$

5. $D^2 y(x) = y(x)$, $y(0) = 2$,
$Dy(0) = 2$, $y = 2e^x$

6. $D^2 y(x) = -y(x)$, $y(0) = 0$,
$Dy(0) = 1$, $y = \sin x$

7. $y' = \sec^2 x$, $y(0) = 0$, $y = \tan x$

8. $y' = -\csc^2 x$, $y(\pi/2) = 0$,
$y = \cot x$

9. $y' = e^{-x}$, $y(0) = -1$, $y = -e^{-x}$

10. $y' = 1/x$, $y(1) = 1$, $y = \ln x$

Explicit and Implicit Solutions.
Identify the given solution as *implicit* or *explicit*. If *implicit*, then solve for y in terms of x by college algebra methods.

11. $y = x + \sin x$

12. $y = x + \sin x$

13. $2y + x^2 + x + 1 = 0$

14. $x - 2y + \sin x + \cos x = 0$

15. $y = e^{\pi}$

16. $e^y = \pi$

17. $e^{2y} = \ln(1 + x)$

18. $\ln|1 + y^2| = e^x$

19. $\tan y = 1 + x$

20. $\sin y = (x - 1)^2$

Tables and Explicit Equations.
For the given explicit equation, make a table of values $x = 0$ to $x = 1$ in steps of 0.2.

21. $y = x^2 - 2x$

22. $y = x^2 - 3x + 1$

23. $y = \sin \pi x$

24. $y = \cos \pi x$

25. $y = e^{2x}$

26. $y = e^{-x}$

27. $y = \ln(1 + x)$

28. $y = x \ln(1 + x)$

Tables and Approximate Equations.
Make a table of values $x = 0$ to $x = 1$ in steps of 0.2 for the given approximate equation. Identify precisely the *recursion* formulas applied to obtain the next table pair from the previous table pair.

29. $y(x + 0.2) \approx y(x) + 0.2(1 - y(x))$,
$y(0) = 1$

30. $y(x + 0.2) \approx y(x) + 0.2(1 + y(x))$,
$y(0) = 1$

31. $y(x + 0.2) \approx y(x) + 0.2(x - y(x))$,
$y(0) = 0$

32. $y(x + 0.2) \approx y(x) + 0.2(2x + y(x))$,
$y(0) = 0$

33. $y(x + 0.2) \approx y(x) + 0.2(\sin x + xy(x))$, $y(0) = 2$

34. $y(x + 0.2) \approx y(x) + 0.2(\sin x - x^2 y(x))$, $y(0) = 2$

35. $y(x + 0.2) \approx y(x) + 0.2(e^x - 7y(x))$,
$y(0) = -1$

36. $y(x + 0.2) \approx y(x) + 0.2(e^{-x} - 5y(x))$, $y(0) = -1$

37. $y(x + 0.2) \approx y(x) + 0.1(e^{-2x} - 3y(x))$, $y(0) = 2$

38. $y(x + 0.2) \approx y(x) + 0.2(\sin 2x - 2y(x))$, $y(0) = 2$

Hand Graphing. Make a graphic by hand on engineering paper, using 6 data points. Cite the divisions assigned horizontally and vertically. Label the axes and the center of coordinates. Supply one sample hand computation per graph. Employ a computer program or calculator to obtain the data points.

39. $y = 3x$, $x = 0$ to $x = 1$.

40. $y = 5x^3$, $x = 0$ to $x = 1$.

41. $y = 2x^5$, $x = 0$ to $x = 1$.

42. $y = 3x^7$, $x = 0$ to $x = 1/2$.

43. $y = 2x^4$, $x = 0$ to $x = 1$.

44. $y = 3x^6$, $x = 0$ to $x = 1$.

45. $y = \sin x$, $x = 0$ to $x = \pi/4$.

46. $y = \cos x$, $x = 0$ to $x = \pi/4$.

47. $y = \dfrac{x+1}{x+2}$, $x = 0$ to $x = 1$.

48. $y = \dfrac{x-1}{x+1}$, $x = 0$ to $x = 1$.

49. $y = \ln(1 + x)$, $x = 0$ to $x = 1$.

50. $y = \ln(1 + 2x)$, $x = 0$ to $x = 1$.

1.4 Direction Fields

The **method of direction fields** is a graphical method for displaying
the general shape and behavior of solutions to $y' = f(x, y)$. It persists as
a fundamental topic because it *does not require solving the differential
equation $y' = f(x, y)$* . The uniform grid method and the isocline method
are introduced, for computer and hand construction of direction fields.

What's a Direction Field?

L. Euler (1707–1783) discovered a way to draw a graphic showing the
behavior of all solutions to a given differential equation, *without solving
the equation*. The graphic is built from a grid of points arranged on a
graph window. Paired with each grid point is a line segment centered on
the grid point. The line segments are non-overlapping.

The graph window plus pairs of grid points and line segments is called a
direction field, provided the line segment at grid point (x_0, y_0) coincides
with the tangent line to the solution of the initial value problem

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases}$$

This means that $y'(x_0)$ equals the slope of the line segment. We don't
have to know a formula for $y(x)$, because $y'(x_0)$ can be computed from
its equivalent formula $y'(x_0) = f(x_0, y_0)$, a number that depends only
on the grid point (x_0, y_0) and the function $f(x, y)$. Euler's intent is to
replace the differential equation model $y' = f(x, y)$ by a graphical model,
a direction field.

$$y' = f(x, y)$$

DE model

Graphical Model
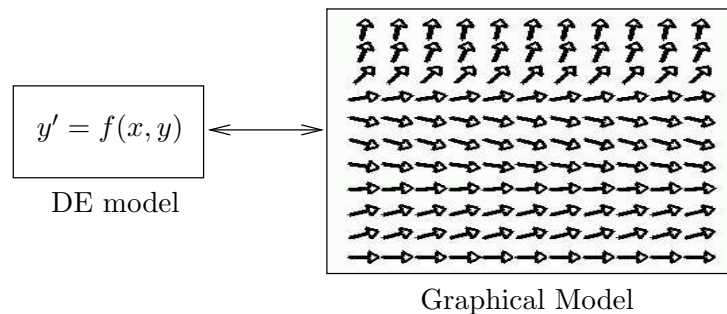
Figure 7. Model Replacement.

A differential equation model $y' = f(x, y)$ is replaced by a direction field model,
a graphic consisting of pairs of grid points and line segments. A direction
field **line segment** is represented by an arrow, to show the direction of the
tangent. A **grid point** is the center point of an arrow shaft. Model replacement
means the differential equation model is tossed aside and we deal only with the
graphical model.

Solution Curves and Direction Fields

Euler's visualization idea begins with the direction field, drawn for some graph window, with pairs of grid points and line segments dense enough to cover most of the white space in the graph window. The *theory* used in Euler's idea consists of a short list of facts:

1. Solutions of $y' = f(x, y)$ don't cross.

2. A tangent to an edge-to-edge solution $y(x)$ nearly matches tangents to nearby direction field segments.

3. Direction field segments are solutions of $y' = f(x, y)$, to pixel resolution.

Details 1: If solutions $y_1(x)$, $y_2(x)$ cross at $x = x_0$, then let $y_0 = y_1(x_0) = y_2(x_0)$ and consider the initial value problem

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases}$$

We assume solutions to all such initial value problems are *unique*. This implies $y_1(x) = y_2(x)$ for $|x - x_0|$ small. Hence crossings are impossible. The analysis implies that two solutions which touch must coalesce.

Direction field segments, representing solution curves, must be constructed not to touch each other. Edge-to-edge solution curves cannot cross a direction field segment, but they may coincide with a direction field segment, to pixel resolution.

Details 2: Tangent vectors for $\vec{\mathbf{r}} = x\vec{i} + y(x)\vec{j}$ are drawn from $\vec{\mathbf{r}}' = \vec{i} + y'(x)\vec{j} = \vec{i} + f(x, y)\vec{j}$. Continuity of f implies that the vector $\vec{\mathbf{r}}'$ is to pixel resolution identical for all (x, y) sufficiently close to a grid point (x_0, y_0). This is why an edge-to-edge solution passes grid points with tangent vector *nearly* matching the tangent vector of nearby segments.

Details 3: Each segment is a tangent line $y = y_0 + m(x - x_0)$, constructed with slope $m = y'(x_0)$. It approximates the curve $y(x)$ local to the contact point (x_0, y_0). Graphically, a short tangent line coalesces with the curve near the contact point, to pixel resolution.

The tangent line approximation is called **Euler's approximation**. Correct pronunciation is **Oiler**. To make the audience giggle, pronounce it **Yuler**.

Rules for Drawing Threaded Solutions

A direction field graphic replaces all the information supplied by the equation $y' = f(x, y)$. The equation is tossed aside and not used.

Visualization of all solutions involves drawing a small number of edge-to-edge solutions $y(x)$ onto the direction field graph window. We will use just two abbreviated rules (see **1** and **2** in Table 4).

1.   Solutions don't cross.

2.   Nearby tangents nearly match.

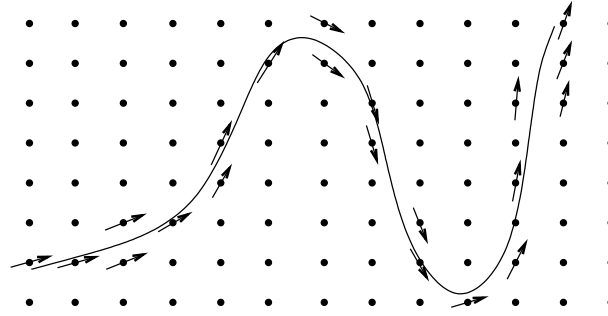**Table 4.   Rules for Drawing Edge-to-Edge Solutions**

Figure 8.   Threading a Solution Edge-to-Edge.

Shown in Figure 8 is a threaded solution curve for $y' = f(x, y)$ plus nearby grid points and relevant line segments (arrows). The solution threads its way through the direction field, matching tangents at nearby grid points. Arrows that touch a threaded curve must coalesce with the curve (solutions don't cross).
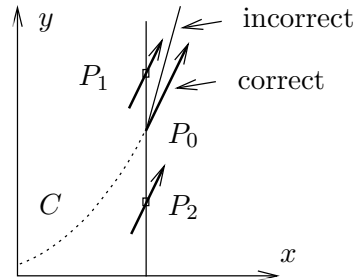

Figure 9.   Threading Rules.

Solution curve C threads from the left and meets a line at point P_0. The line contains two nearby grid points P_1, P_2. The tangent at P_0 must nearly match direction field arrows at P_1, P_2.

Tangent Matching Explained.

The slopes of the tangents in Figure 9 are given by $y' = f(x, y)$. For points $P_1 = (x_1, y_1)$, $P_0 = (x_0, y_0)$ and $P_2 = (x_2, y_2)$, the slopes are $f(x_1, y_1)$, $f(x_0, y_0)$, $f(x_2, y_2)$. If the points P_0, P_1, P_2 are close, then continuity of f implies all three slopes are *nearly equal*.

How to Construct a Direction Field Graphic

Window   Invent the graph x-range and y-range.   **1**

Grid   Plot a uniform grid of N grid points within the graph window. Invent N to populate the graphical white space, $N \approx 50$ for hand work.   **2**

Field   Draw at each grid point (x^*, y^*) a short tangent vector $\epsilon \vec{T}$, where $\vec{T} = \vec{i} + f(x^*, y^*)\vec{j}$.   **3**

Threaded Solutions   Draw additional threaded solutions on long x-intervals into the remaining white space of the graphic.   **4**

Construction Notes.

$\boxed{1}$ The window should include all significant equilibrium solutions, that is, solutions $y = $ constant of $y' = f(x, y)$, which plot as horizontal lines. Physically interesting initial conditions $(x_0, y(x_0))$ should be added.

$\boxed{2}$ The isocline method might also be used to select grid points. For details on both methods, see the next subsection.

$\boxed{3}$ The arrow shaft is a **replacement curve** for the solution of $y' = f(x, y)$ through grid point (x^*, y^*) on a small x-interval, called a **lineal element**.

$\boxed{4}$ Threading is educated guesswork, discussed above, in Figures 8 and 9. If possible, choose (x_0, y_0) on the left window edge, then thread the solution until it exits the window top, bottom or right.

Direction fields are studied again in Chapter 10, page 694, in connection with phase portraits of two-dimensional systems of differential equations.

Two Methods for Selecting Grid Points

There are two standard methods for selecting grid points, called the **uniform grid method** and the **isocline grid method**. The methods may be combined in some applications.

Uniform Grid   Two positive increment parameters n and m are supplied along with a graph window $a \leq x \leq b$, $c \leq y \leq d$. Hand work usually starts with $n = m = 11$; computer software starts with $n = m = 21$.

The nm grid points are defined for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ by the equations $x_i = a + (b - a)(i - 1)/(n - 1)$, $y_j = c + (d - c)(j - 1)/(m - 1)$.

Isocline Grid   A graph window $a \leq x \leq b$, $c \leq y \leq d$ is given plus a list of invented slopes $M_1, \ldots, M_p$ for the lineal elements.

To define the grid points, select the number $n > 0$ of grid points to be drawn on each isocline. Construct n equally-spaced horizontal lines (or vertical lines). Define grid points as intersections of the lines with all the implicit curves $f(x, y) = M_\ell$, $\ell = 1, \ldots, p$.

Along the implicit curve $f(x, y) = M_\ell$, within the graph window, mark each grid point and draw a lineal element, each element of exactly the same slope M_ℓ, for $\ell = 1, \ldots, p$.

The two methods are applied in Examples 27 and 28, page 45. Illustrated for the isocline method are possibilities such as graph window clipping and fine-tuning of the slopes to allow the grid points to fill the window.

Grid points in the *isocline method* are intersections of equally-spaced lines with the implicit curves $f(x, y) = M_\ell$. Lineal elements sketched along this curve all have slope M_ℓ and therefore they can be drawn with reduced effort.

How to Make Lineal Elements

A lineal element is a line segment centered at a grid point. They should not touch, because they represent, to pixel resolution, non-crossing solution curves on a short x-interval. Choose H to be not greater than the minimum distance between pairs of grid points. Initially, one can guess the value of H, then adjust the value after seeing the result. Define

$$h = \frac{H}{2\sqrt{1 + |f(x_0, y_0)|^2}}.$$

Then a lineal element of length H is defined by the midpoint (x_0, y_0) and the two endpoints $(x_0 - h, y_0 - hM)$ and $(x_0 + h, y_0 + hM)$, where $M = f(x_0, y_0)$.

This choice insures lineal elements do not touch. It is possible to erase the line segment to the left or right of the grid point without losing much information. Arrow heads can be added to show the tangent direction.

Examples

25 Example (Window and Grid) Choose a graph window for the differential equation $y' = y^2(2 - y)(1 + y)$ which includes the equilibrium solutions. Draw a 5×5 uniform grid on the graph window and plot the equilibrium solutions. Do not draw the direction field nor threaded solutions.

Solution: Let $f(x, y) = y^2(2 - y)(1 + y)$. Then $y = k$ is a constant solution of $y' = f(x, y)$ exactly when $0 = k^2(2 - k)(1 + k)$. The values $k = -1, 0, 2$ give horizontal lines $y = -1$, $y = 0$, $y = 2$. These lines are called equilibrium solutions; they are constant solutions of the differential equation. Accordingly, a graph window containing the equilibria is $-3 \le x \le 3$, $-2 \le y \le 3$. The 25 grid points are obtained by the formulas $x_k = -3 + 6(k/4)$, $k = 0, \ldots, 4$ and $y_j = -2 + 5(j/4)$, $j = 0, \ldots, 4$. The plot is done by hand. A computer plot appears in Figure 10.
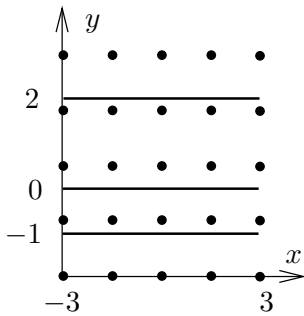
Figure 10. A graph window with uniform grid and equilibria.
Three equilibrium solutions $y = -1$, $y = 0$, $y = 2$ appear plus 25 grid points on the graph window $|x| \leq 3$, $-2 \leq y \leq 3$.

26 Example (Threading a Solution) Starting at the black dots in the direction field graphic of Figure 11, thread three solution curves.
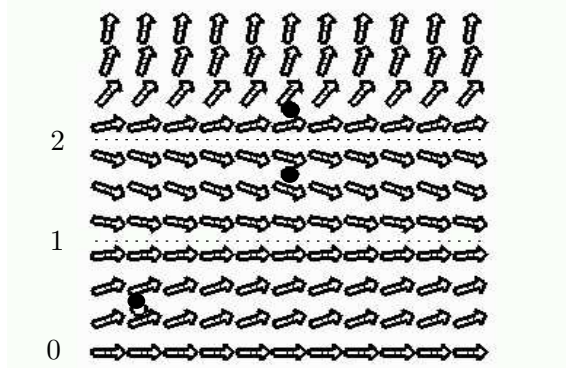

Figure 11. A direction field.
A field for the differential equation $y' = y(2 - y)(1 - y)$ is plotted on graph window $0 \leq x \leq 3$, $0 \leq y \leq 3$. The black dots are at $(0.25, 0.4)$, $(1.5, 2.25)$ and $(1.5, 1.65)$.

Solution: A plot appears in Figure 12. A threaded solution matches its tangents with nearby lineal elements of the direction field in Figure 11; see page 41 for an explanation. Each threaded curve represents a solution of the differential equation through the given dot on the entire interval $0 \leq x \leq 3$, whereas the lineal elements represent solutions through the grid point on a very short x-interval.
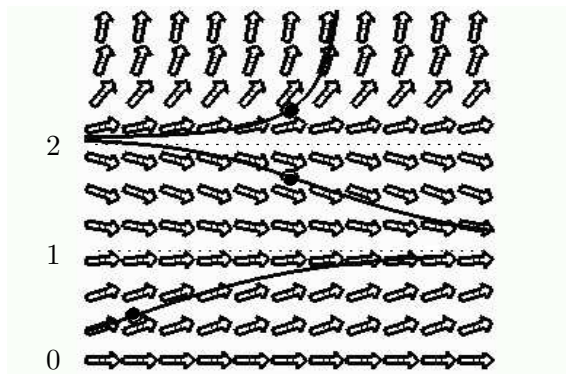
Figure 12. Threaded solutions.
The graph window is $0 \leq x \leq 3$, $0 \leq y \leq 3$. Threaded curves cannot cross equilibrium solutions $y = 0$, $y = 1$ and $y = 2$.

27 Example (Uniform Grid Method) Make a direction field of 11×11 points for $y' = x + y(1 - y)$ on $-1 \leq x \leq 1$, $-2 \leq y \leq 2$.

Solution: Let $f(x, y) = x + y(1 - y)$. The 121 grid points are the pairs (x, y)

where $x = -1$ to 1 in increments of 0.2 and $y = -2$ to 2 in increments of 0.4. The minimum distance between grid points is $H = 0.2$.

We will generate the endpoints of the lineal element at $x_0 = -0.4$, $y_0 = 1.6$. It will be shown that the first endpoint is $(-0.34076096, 1.5194349)$. This point can be located from (x_0, y_0) by traveling distance $H/2$ at slope $M = -1.36$.

$M = f(x_0, y_0)$	The line segment slope for Euler's rule.
$\quad = x_0 + y_0(1 - y_0)$	Apply $f(x, y) = x + y(1 - y)$.
$\quad = -1.36,$	Use the first point $x_0 = -0.4$, $y_0 = 1.6$.
$h = \dfrac{H}{2\sqrt{1 + M^2}}$	Apply the formula $h = (H/2)/\sqrt{1 + f(x_0, y_0)^2}$.
$\quad = 0.059239045,$	Use $H = 0.2$ and $f(x_0, y_0) = M = -1.36$.
$X = x_0 + h$	Compute the x-coordinate of the second point.
$\quad = -0.34076096$	Use $x_0 = -0.4$ and $h = 0.059239045$.
$Y = y_0 + hf(x_0, y_0)$	Compute the y-coordinate of the second point.
$\quad = 1.5194349$	Use values $y_0 = 1.6$, $f(x_0, y_0) = M = -1.36$, $h = 0.059239045$.

The second endpoint $(-0.459239045, 1.6805651)$, at distance $H/2$ from the grid point, in the opposite direction, can be found by minor changes to the above calculation. Automation of this process is necessary because 121 such calculations are required. Some basic `maple` code appears below which computes the 121 pairs of points for the direction field, then plots a replica of the field. The graphic appears in Figure 13. The code adapts to numerical laboratories like `matlab`, `octave` and `scilab`, which may or may not have a suitable direction field library, depending on the version.
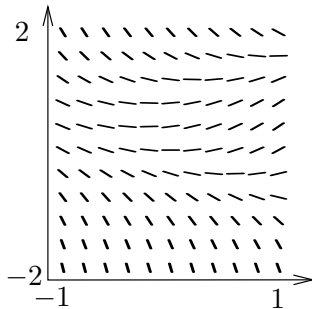

Figure 13. Direction field for the equation $y' = x + y(1 - y)$.

The uniform grid method is used on graph window $-1 \le x \le 1$, $-2 \le y \le 2$. There are 121 grid points.

```
a:=-1:b:=1:c:=-2:d:=2:n:=11:m:=11:
H:=(b-a)/(n-1):K:=(d-c)/(m-1):HH:=0.15:
f:=(x,y)->x+y*(1-y): X:=t->a+H*(t-1):Y:=t->c+K*(t-1):P:=[]:
for i from 1 to n do for j from 1 to m do
  x0:=X(i):y0:=Y(j):M:=evalf(f(x0,y0)):
  h:=evalf((HH/2)/sqrt(1+M^2)):
  Seg:=[[x0-h,y0-h*M],[x0+h,y0+h*M]]:
  if (P = []) then P:=Seg: next: fi: P:=P,Seg:
od:od:
opts:=scaling=constrained,color=black,thickness=3,axes=boxed;
plot([P],opts);
```

Versions of `maple` since V 5.1 have a `DEtools` package which simplifies the process of making a direction field. In `mathematica`, a similar command exists.

```
with(DEtools):  de:=diff(y(x),x)=x+y(x)*(1-y(x)):  # Maple
opts:=arrows=LINE,dirgrid=[11,11];
DEplot(de,y(x),x=-1..1,y=-2..2,opts);
```

```
<< Graphics\PlotField.m # Mathematica
PlotVectorField[1,x+y (1-y),x,-1,1,y,-2,2]
```

Resources for computer-assisted direction fields with interactive threaded solutions include `Maple`, `Mathematica`, `Matlab`. The 2014 `Matlab` interface uses open-source computer algebra system `Maxima`. Each system has a steep learning curve, well worth the investment time.

28 Example (Isocline Method) Make a direction field by the isocline method for the differential equation $y' = x + y(1 - y)$ on $-1 \le x \le 1$, $-1 \le y \le 2$.

Solution: Let $f(x, y) = x + y(1-y)$ and let M denote the slope of a replacement lineal element. The isoclines are defined by $f(x, y) = M$. It has the standard equation $(y - 1/2)^2 = x - M + 1/4$, which is a parabola with center $(M - 1/4, 1/2)$ opening to the right. The algebra details:

$x + y(1 - y) = M$	The equation $f(x, y) = M$ expanded.
$y^2 - y = x - M$	Multiply by -1 and move $-x$ to the right side.
$y^2 - y + \frac{1}{4} = x - M + \frac{1}{4}$	Apply *square completion*: add the square of half the coefficient of y to both sides.
$(y - \frac{1}{2})^2 = x - M + \frac{1}{4}$	Write the left side as a perfect square. It has the form of the standard curve library equation $Y^2 = X$. See page 880.

The basic requirement for slope M selection is that the set of grid points obtained below fills the white space of the graph window. Briefly, some portion of each parabola has to intersect the graph window. By experiment, the slopes M to be used in the isocline method will be selected as $M = 1/4 + (-3)$ to $M = 1/4 + (1)$ in increments of 0.2 to identify 21 isoclines:

Isocline Equation	Slope M
$(y - 1/2)^2 = x - (-3)$	$0.25 + (-3)$
$(y - 1/2)^2 = x - (-2.8)$	$0.25 + (-3) + 0.2$
$\vdots$	$\vdots$
$(y - 1/2)^2 = x - (1)$	$0.25 + (-3) + 4.0$

To define the grid points, let $y = -1$ to 2 in increments of 0.3 to make 11 horizontal lines. The intersections of these 11 lines with the 21 parabolas define at least 100 grid points inside the graph window. It is possible to graph rapidly the 21 parabolas, because they are translates of the standard parabola $Y^2 = X$.

The replacement lineal elements on each parabola are sketched rapidly as follows. Using pencil and paper, graph accurately the first lineal element on the isocline curve, using associated slope M. Rotate the paper until the lineal element is vertical. Draw additional lineal elements at the remaining grid points of

the isocline curve as vertical lines. Accuracy improves with the use of a drawing easel, T-square and triangle.

A computer graphic is shown in Figure 14 which closely resembles a hand-made graphic. Compare it to the uniform grid method graphic in Figure 13, page 46.
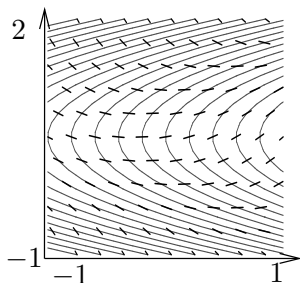

Figure 14. Direction field for the differential equation $y' = x + y(1 - y)$**.**

The isocline method is applied using graph window $-1 \leq x \leq 1$, $-1 \leq y \leq 2$. Parabolas are isoclines. Grid points are intersections of isoclines with equally-spaced horizontal lines.

The `maple` code that produced Figure 14 is included below to show the machine equivalents of a hand computation. The ordering of events in the code is reversed: the grid points and lineal elements are drawn first, then the isocline curves, finally the two graphics are superimposed. A key detail is solving $f(x_0, y_0) = M$ for x_0 to locate a grid point (x_0, y_0) (in this application, $x_0 = M + y_0(y_0 - 1)$). The clipping test $a \leq x_0 \leq b$ is applied to keep lineal elements inside the graph window.

```
with(plots):
 a:=-1:b:=1:n:=21:c:=-1:d:=2:m:=11:
 H:=0.1:F:=-3:G:=1:f:=unapply(x+y*(1-y),(x,y)):
 Slope:=unapply(F+1/4+(G-F)*(t-1)/(n-1),t):
 Y:=unapply(c+(d-c)*(t-1)/(m-1),t): P:=[]:
 for j from 1 to n do
     M:=Slope(j):h:=evalf(H*0.5/sqrt(1+M^2)):
     for k from 1 to m do
      y0:=Y(k): x0:=solve(f(x,y0)=M,x);
      if x0<a or x0>b then next: fi:
      Seg:=[[x0-h,y0-h*M],[x0+h,y0+h*M]]:
      if P=[] then P:=Seg: next: fi:
      P:=P,Seg:
 od:od:
 opts:=color=BLACK,thickness=3,axes=none,scaling=constrained:
 Window:=x=a..b,y=c..d:
 Plot1:=plot([P],Window,opts):
 eqs:={seq((y-0.5)^2=x-Slope(j)+1/4,j=1..n)}:
 Plot2:=implicitplot(eqs,Window):
 display([Plot1,Plot2]);
```

Exercises 1.4

Window and Grid. Find the equilibrium solutions, then determine a graph window which includes them and construct a 5×5 uniform grid. Follow Example 25.

1. $y' = 2y$

2. $y' = 3y$

3. $y' = 2y + 2$

4. $y' = 3y - 2$

5. $y' = y(1 - y)$

6. $y' = 2y(3 - y)$

7. $y' = y(1 - y)(2 - y)$

8. $y' = 2y(1 - y)(1 + y)$

9. $y' = 2(y - 1)(y + 1)^2$

10. $y' = 2y^2(y - 1)^2$

11. $y' = (x + 1)(y + 1)(y - 1)y$

12. $y' = 2(x + 1)y^2(y + 1)(y - 1)^2$

13. $y' = (x + 2)y(y - 3)(y + 2)$

14. $y' = (x + 1)y(y - 2)(y + 3)$

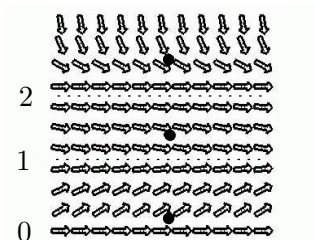Threading Solutions. Each direction field below has window $0 \le x \le 3$, $0 \le y \le 3$. Start each threaded solution at a black dot and continue it left and right across the field. Dotted horizontal lines are equilibrium solutions. See Example 26.

15.

16.

17.

18.

19.

20.

21.

22.
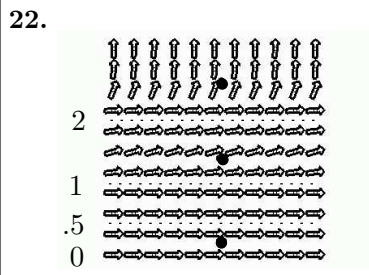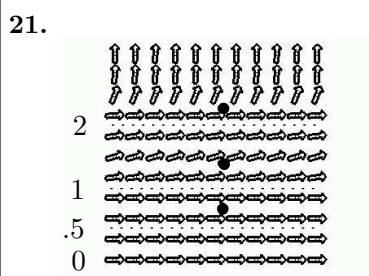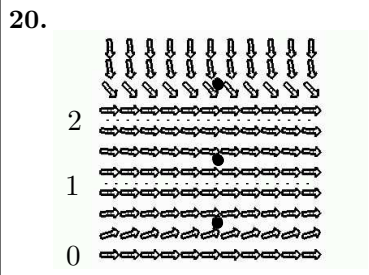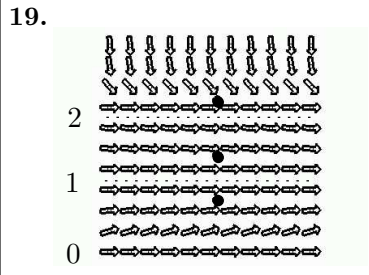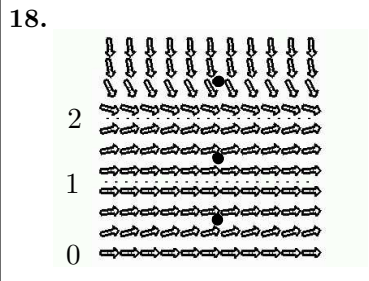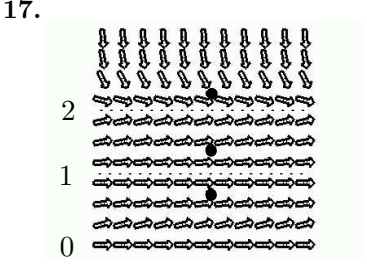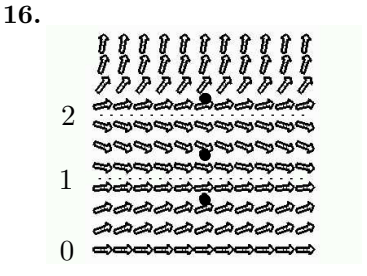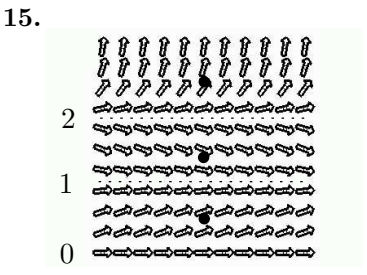
23.
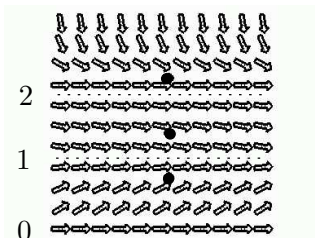
24.

Uniform Grid Method. Apply the uniform grid method as in Example 27, page 45 to make a direction field of 11×11 grid points for the given differential equation on $-1 \le x \le 1$, $-2 \le y \le 2$. If using a computer program, then use about 20×20 grid points.

25. $y' = 2y$

26. $y' = 3y$

27. $y' = 1 + y$

28. $y' = 2 + 3y$

29. $y' = x + y(2 - y)$

30. $y' = x + y(1 - 2y)$

31. $y' = 1 + y(2 - y)$

32. $y' = 1 + 2y(2 - y)$

33. $y' = x - y$

34. $y' = x + y$

35. $y' = y - \sin(x)$

36. $y' = y + \sin(x)$

Isocline Method. Apply the isocline method as in Example 28, page 47 to make a direction field of about 11×11 points for the given differential equation on $0 \le x \le 1$, $0 \le y \le 2$. Computer programs are used on these kinds of problems to find grid points as intersections of isoclines and lines. Graphics are expected to be done by hand. Large white spaces in the graphic should be filled by choosing a richer set of slopes.

37. $y' = x - y^2$

38. $y' = 2x - y^2$

39. $y' = 2y/(x + 1)$

40. $y' = -y^2/(x + 1)^2$

41. $y' = \sin(x - y)$

42. $y' = \cos(x - y)$

43. $y' = xy$

44. $y' = x^2 y$

45. $y' = xy + 2x$

46. $y' = x^2 y + 2x^2$

1.5 Phase Line and Bifurcation Diagrams

Technical publications may use special diagrams to display **qualitative information** about the equilibrium points of the differential equation

$$(1) \qquad\qquad y'(x) = f(y(x)).$$

The right side of this equation is independent of x, hence there are no external control terms that depend on x. Due to the lack of external controls, the equation is said to be **self-governing** or **autonomous**.

A **phase line diagram** for the autonomous equation $y' = f(y)$ is a line segment with labels **sink**, **source** or **node** (definitions below), one mark and label for each root y of $f(y) = 0$, i.e., each equilibrium; see Figure 15. A phase line diagram summarizes the contents of a direction field and all equilibrium solutions. It is used to draw threaded curves across the graph window, producing a **phase portrait** for $y' = f(y)$.

The function f must be one-signed on the interval between adjacent equilibrium points, because $f(y) = 0$ means y is an equilibrium point. For this reason, a sign $+$ or $-$ is written on a phase line diagram between each pair of adjacent equilibria.

$$
\begin{array}{ccccc}
- & + & + & - & - \\
y_0 & y_2 & y_1 & y_3 \\
\text{source} & \text{node} & \text{sink} & \text{node}
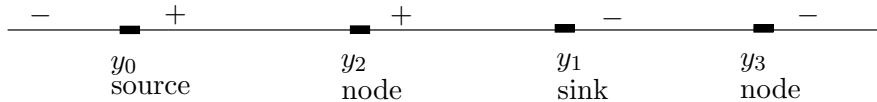\end{array}
$$

Figure 15. A phase line diagram.

The labels **sink**, **source**, **node** are borrowed from the theory of fluids and they have the following **special definitions**:[5]

Sink $y = y_0$	The equilibrium $y = y_0$ *attracts* nearby solutions at $x = \infty$: for some $H > 0$, $|y(0) - y_0| < H$ implies $|y(x) - y_0|$ decreases to 0 as $x \to \infty$.
Source $y = y_1$	The equilibrium $y = y_1$ *repels* nearby solutions at $x = \infty$: for some $H > 0$, $|y(0) - y_1| < H$ implies that $|y(x) - y_1|$ increases as $x \to \infty$.
Nodes $y = y_2$ and $y = y_3$	The equilibrium $y = y_2$ is neither a sink nor a source.

---

[5]It is for the reader's geometric intuition that the current text section requires monotonic behavior in the definition of a sink. In applied literature a sink is defined by $\lim_{x\to\infty} |y(x) - y_0| = 0$, an easy transition for most readers, although unnecessarily abstract. See page 55 for definitions of **attracting** and **repelling** equilibria.

Drawing Phase Portraits

A phase line diagram is used to draw a **phase portrait** of threaded solutions and equilibrium solutions by using the three rules below.

1. Equilibrium solutions are horizontal lines in the phase diagram.

2. Threaded solutions of $y' = f(y)$ don't cross.[6] In particular, they don't cross equilibrium solutions.

3. A threaded non-equilibrium solution that starts at $x = 0$ at a point y_0 must be increasing if $f(y_0) > 0$, and decreasing if $f(y_0) < 0$.

To justify **3**, let $y_1(x)$ be a solution with $y_1'(x) = f(y_1(x))$ either positive or negative at $x = 0$. If $y_1'(x_1) = 0$ for some $x_1 > 0$, then let $c = y_1(x_1)$ and define equilibrium solution $y_2(x) = c$. Then solution y_1 crosses the equilibrium solution y_2 at $x = x_1$, violating rule **2**.
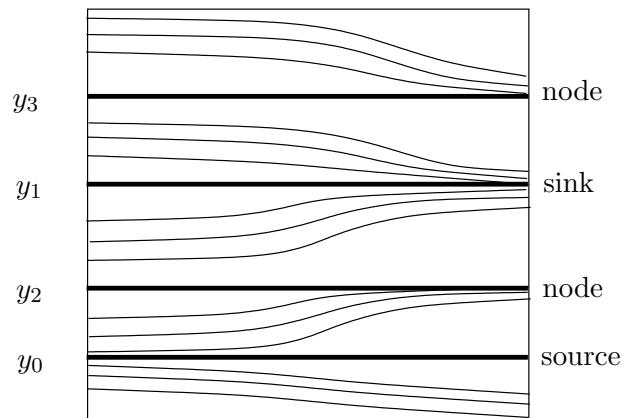

Figure 16. A phase portrait for an autonomous equation $y' = f(y)$.
The graphic is drawn directly from phase line diagram Figure 15, using rules **1**, **2**, **3**. While not a replica of an accurately constructed computer graphic, the general look of threaded solutions is sufficient for intuition. Labels **source**, **sink**, **node** are essential. Alternate labels: **spout**, **funnel**, **node**.

Figure 15 shows that classifications **source, sink, node** (or **spout, funnel, node**) can be decided from the signs of $f(y)$ left and right of an equilibrium point.

---

[6]In normal applications, solutions to $y' = f(y)$ will not cross one another. Technically, this requires uniqueness of solutions to initial value problems, satisfied for example if f and f' are continuous, because of the Picard-Lindelöf theorem.

Table 5.   Equilibria Classification by Signs of $f(y)$

Classification	Sign of $f(y)$ left	Sign of $f(y)$ right
Source [Spout]	MINUS	PLUS
Sink [Funnel]	PLUS	MINUS
Node	PLUS	PLUS
Node	MINUS	MINUS

Drain and Spout

In fluids, **source** means fluid is created and **sink** means fluid is lost. A memory device for these concepts is the kitchen water spout, which is the *source*, and the kitchen drain, which is the *sink*.

Figure 17.   A source or a spout.
A water **spout** from a kitchen faucet or a spray-can is a **source**. Pencil traces in a figure represent flow lines in the fluid.
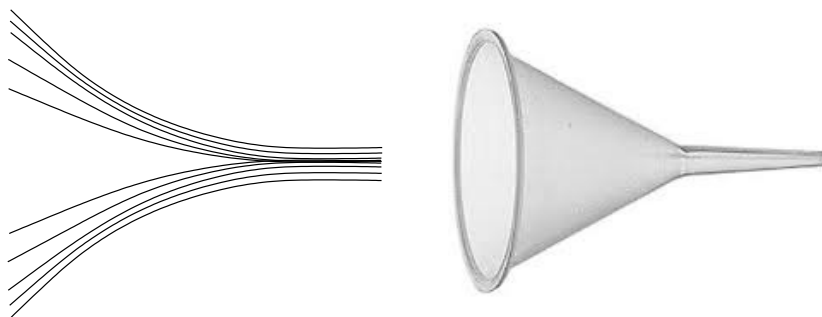

Figure 18.   A sink or a funnel.
A **funnel** rotated 90 degrees has the shape of a **sink**. A drain in the kitchen sink has the same geometry. The lines drawn in a funnel figure can be visualized as traces of flow lines or dust particles in the fluid, going down the drain.
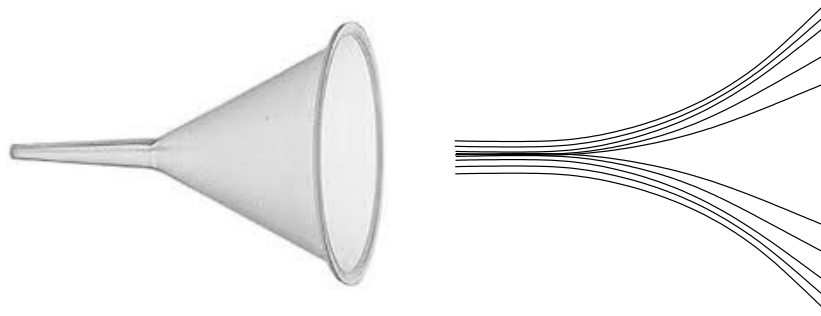
Figure 19. Video replay in reverse time.
A video of a funnel or sink played backwards looks like a source or spout.

Stability Test

The terms **stable equilibrium** and **unstable equilibrium** refer to the predictable plots of nearby solutions. The term **stable** means that solutions that start near the equilibrium will stay nearby as $x \to \infty$. The term **unstable** means *not stable*. Therefore, a sink is stable and a source is unstable.

Definition 7 (Stable Equilibrium)
An equilibrium y_0 of $y' = f(y)$ is **stable** provided for given $\epsilon > 0$ there exists some $H > 0$ such that $|y(0) - y_0| < H$ implies solution $y(x)$ exists for $x \geq 0$ and $|y(x) - y_0| < \epsilon$ for all $x \geq 0$.

The solution $y = y(0)e^{kx}$ of the equation $y' = ky$ exists for $x \geq 0$. Properties of exponentials justify that the equilibrium $y = 0$ is a sink for $k < 0$, a source for $k > 0$ and just stable for $k = 0$.

Definition 8 (Attracting and Repelling Equilibria)
An equilibrium $y = y_0$ is **attracting** provided $\lim_{x \to \infty} y(x) = y_0$ for all initial data $y(0)$ with $0 < |y(0) - y_0| < h$ and $h > 0$ sufficiently small. An equilibrium $y = y_0$ is **repelling** provided $\lim_{x \to -\infty} y(x) = y_0$ for all initial data $y(0)$ with $0 < |y(0) - y_0| < h$ and $h > 0$ sufficiently small.

The **stability test** below in Theorem 3 is motivated by the vector calculus results **Div**(**P**) < 0 for a sink and **Div**(**P**) > 0 for a source, where **P** is the velocity field of the fluid and **Div** is divergence. Justification is postponed to page 60.

Theorem 3 (Stability and Instability Conditions)
Let f and f' be continuous. The equation $y' = f(y)$ has a *sink* at $y = y_0$ provided $f(y_0) = 0$ and $f'(y_0) < 0$. An equilibrium $y = y_1$ is a *source*

provided $f(y_1) = 0$ and $f'(y_1) > 0$. There is *no test* when f' is zero at an equilibrium. The no-test case can sometimes be decided by an additional test:

(a) Equation $y' = f(y)$ has a *sink* at $y = y_0$ provided $f(y)$ changes sign from positive to negative at $y = y_0$.

(b) Equation $y' = f(y)$ has a *source* at $y = y_0$ provided $f(y)$ changes sign from negative to positive at $y = y_0$.

Phase Line Diagram for the Logistic Equation

The model logistic equation $y' = (1 - y)y$ is used to produce the phase line diagram in Figure 20. The logistic equation is discussed on page 6, in connection with the Malthusian population equation $y' = ky$. The letters S and U are used for a stable sink and an unstable source, while N is used for a node. For computational details, see Example 30, page 58.
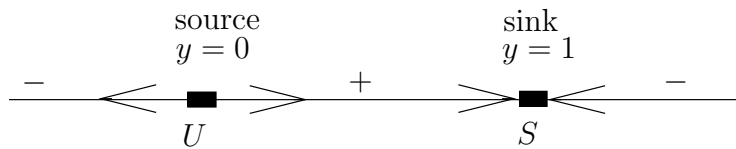

Figure 20. A phase line diagram for $y' = (1 - y)y$.
The equilibrium $y = 0$ is an unstable source (a spout) and equilibrium $y = 1$ is a stable sink (a funnel).

Arrowheads are used to display the **repelling** or **attracting** nature of the equilibrium.

Direction Field Plots

A direction field for an autonomous differential equation $y' = f(y)$ can be constructed in two steps.

> **Step 1**. Draw grid points and line segments along the y-axis.

> **Step 2**. Duplicate the y-axis direction field at even divisions along the x-axis.

Duplication is justified because $y' = f(y)$ does not depend on x, which means that the slope assigned to a line segment at grid points $(0, y_0)$ and (x_0, y_0) are identical.

The following facts are assembled for reference:

> **Fact 1**. An equilibrium is a horizontal line. It is *stable* if all solutions starting near the line remain nearby as $x \to \infty$.

Fact 2. Solutions don't cross. In particular, any solution that starts above or below an equilibrium solution must remain above or below.

Fact 3. A solution curve of $y' = f(y)$ rigidly moved to the left or right will remain a solution, i.e., the translate $y(x - x_0)$ of a solution to $y' = f(y)$ is also a solution.

A phase line diagram is merely a summary of the solution behavior in a direction field. Conversely, an independently made phase line diagram can be used to enrich the detail in a direction field.

Fact 3 is used to make additional threaded solutions from an initial threaded solution, by translation. Threaded solutions with turning points are observed to have translations with turning points marching monotonically to the left, or to the right.

Bifurcations

The phase line diagram has a close relative called a **bifurcation diagram**. The purpose of the diagram is to display qualitative information about equilibria, across all equations $y' = f(y)$, obtained by varying physical parameters appearing implicitly in f. In the simplest cases, each parameter change to $f(y)$ produces one phase line diagram and the two-dimensional stack of these phase line diagrams is the bifurcation diagram (see Figure 21).

Fish Harvesting. To understand the reason for such diagrams, consider a private lake with fish population $y(t)$. The population is harvested at rate k fish per year. A suitable sample logistic model is

$$\frac{dy}{dt} = y(4 - y) - k$$

where the constant harvesting rate k is allowed to change. Given some relevant values of k, a biologist would produce corresponding phase line diagrams, then display them by stacking, to obtain a two-dimensional diagram, like Figure 21.
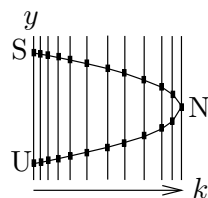

Figure 21. A bifurcation diagram.
The fish harvesting diagram consists of stacked phase-line diagrams.
Legend: U=Unstable, S=Stable, N=node.

In the figure, the vertical axis represents initial values $y(0)$ and the horizontal axis represents the harvesting rate k (axes can be swapped).

The bifurcation diagram shows how the number of equilibria and their classifications *sink*, *source* and *node* change with the harvesting rate.

Shortcut methods exist for drawing bifurcation diagrams and these methods have led to succinct diagrams that remove the phase line diagram detail. The basic idea is to eliminate the vertical lines in the plot, and replace the equilibria **dots** by a curve, essentially obtained by **connect-the-dots**. In current literature, Figure 21 is generally replaced by the more succinct Figure 22.
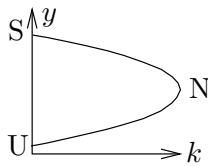
Figure 22. A succinct bifurcation diagram for fish harvesting.
Legend: U=Unstable, S=Stable, N=node.

Stability and Bifurcation Points

Biologists call a fish population *stable* when the fish reproduce at a rate that keeps up with harvesting. Bifurcation diagrams show how to stock the lake and harvest it in order to have a stable fish population.

A point in a bifurcation diagram where stability changes from stable to unstable is called a **bifurcation point**, e.g., label N in Figure 22.

The upper curve in Figure 22 gives the equilibrium population sizes of a stable fish population. Some combinations are obvious, e.g., a harvest of 2 thousand per year from an equilibrium population of about 4 thousand fish. Less obvious is a **sustainable harvest** of about 4 thousand fish with an equilibrium population of about 2 thousand fish, detected from the portion of the curve near the bifurcation point.

Harvesting rates greater than the rate at the bifurcation point will result in **extinction**. Harvesting rates less than this will also result in extinction, if the stocking size is less than the critical value realized on the lower curve in the figure. These facts are justified solely from the phase line diagram, because extinction means all solutions limit to $y = 0$.

Briefly, the lower curve gives the **minimum stocking size** and the upper curve gives the **limiting population** or **carrying capacity**, for a given harvesting rate k on the abscissa.

Examples

29 Example (No Test in Sink–Source Theorem 3) Find an example $y' = f(y)$ which has an unstable node at $y = 0$ and no other equilibria.

Solution: Let $f(y) = y^2$. The equation $y' = f(y)$ has an equilibrium at $y = 0$. In Theorem 3, there is a *no test* condition $f'(0) = 0$.

A computer algebra system can determine $y = 1/(1/y(0) - x)$:

```
dsolve(diff(y(x),x)=y(x)^2,y(x));   maple
ode2('diff(y,x) = y^2,y,x);         Maxima
```

Solutions with $y(0) < 0$ limit to the equilibrium solution $y = 0$, but positive solutions "blow up" before $x = \infty$ at $x = 1/y(0)$. The equilibrium $y = 0$ is an unstable node, that is, it is not a source nor a sink.

The same conclusions are obtained from basic calculus, without solving the differential equation. The reasoning: y' has the sign of y^2, then $y' \geq 0$ implies $y(x)$ increases. The equilibrium $y = 0$ behaves like a source when $y(0) > 0$. For $y(0) < 0$, again $y(x)$ increases, but in this case the equilibrium $y = 0$ behaves like a sink. Accordingly, $y = 0$ is not a source nor a sink, but a node.

30 Example (Phase Line Diagram) Verify the phase line diagram in Figure 23 for the logistic equation $y' = (1 - y)y$, using Theorem 3.
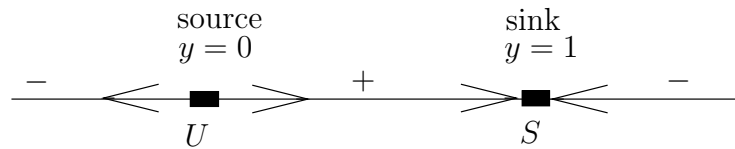

Figure 23.   Phase line diagram for $y' = (1 - y)y$.

Solution: Let $f(y) = (1 - y)y$. To justify Figure 23, there are three steps:

   1. Find the equilibria. Answer: $y = 0$ and $y = 1$.
   2. Find the signs PLUS and MINUS.
   3. Apply Theorem 3 to show $y = 0$ is a source and $y = 1$ is a sink.

The plan is to first compute the equilibrium points.

$(1 - y)y = 0$	Solving $f(y) = 0$ for equilibria.
$y = 0,\ y = 1$	Roots found.

The signs $\boxed{+}$ and $\boxed{-}$ appearing in Figure 20 are labels that mean f is positive or negative on the interval between adjacent equilibria.

A sign of plus or minus is determined by the sign of $f(x)$ for x between equilibria. To justify this statement, suppose both signs occur, $f(x_1) > 0$ and $f(x_2) < 0$. Then continuity of f implies $f(x) = 0$ for a point x between x_1, x_2, which is impossible on an interval free of roots.

The method to determine the signs, plus or minus, then reduces to evaluation of $f(x)$ for an invented sample x chosen between two equilibria, for instance:

$f(-1) = (y - y^2)\big|_{x=-1} = -2$	The sign is MINUS. Chosen was $x = -1$, which is in the interval $-\infty < x < 0$.
$f(0.5) = (y - y^2)\big|_{x=0.5} = 0.25$	The sign is PLUS. Chosen was $x = 0.5$, which is in the interval $0 < x < 1$.
$f(2) = (y - y^2)\big|_{x=2} = -2$	The sign is MINUS. Chosen was $x = 2$, which is in the interval $1 < x < \infty$.

We will apply Theorem 3. The plan is to find $f'(y)$ and then evaluate f' at each equilibrium. An alternative technique is to apply Theorem 3, part **(a)** or **(b)**, which is the method of choice in practise.

$f'(y) = (y - y^2)'$	Find f' from $f(y) = (1 - y)y$.
$\quad = 1 - 2y$	Derivative $f'(y) = \frac{df}{dy}$ found.
$f'(0) = 1$	Positive means it is a *source* (*spout*), by Theorem 3.
$f'(1) = -1$	Negative means it is a *sink* (*funnel*), by Theorem 3.

$y = 1$ Sink or Funnel
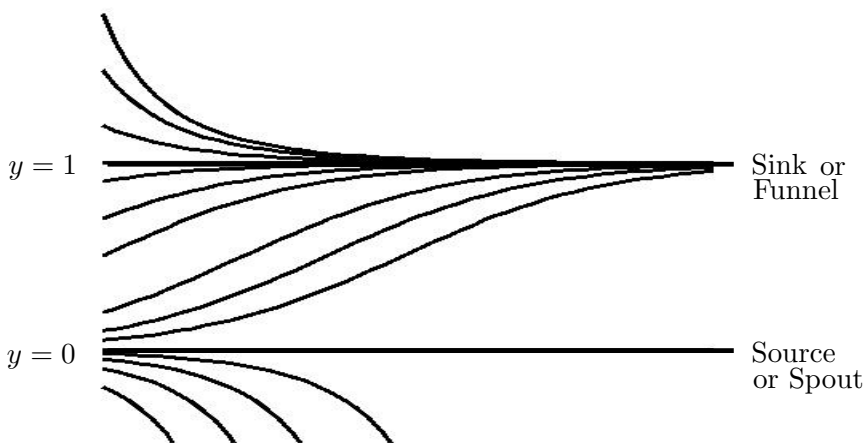
$y = 0$ Source or Spout

Figure 24. Phase portrait for $y' = (1 - y)y$.
Drawn from the phase line diagram of Example 30.

31 Example (Phase Portrait) Justify the phase portrait in Figure 24 for the logistic equation $y' = (1 - y)y$, using the phase line diagram constructed in Example 30.

Solution:

Drawing rules. The phase line diagram contains all essential information for drawing threaded curves. Threaded solutions have to be either horizontal (an equilibrium solution), increasing or decreasing. Optional is representation of turning points.

Translations. Because translates of solutions are also solutions and solutions are unique, then the drawing of an increasing or decreasing threaded curve determines the shape of all nearby threaded curves. There is **no option** for drawing nearby curves!

Explanation. The phase portrait is drawn by moving the phase line diagram to the y-axis of the graph window $0 \leq x \leq 6$, $-0.5 \leq y \leq 2$. The graph window was selected by first including the equilibrium solutions $y = 0$ and $y = 1$, then growing the window after an initial graph. Each equilibrium solution produces a horizontal line, i.e., lines $y = 0$ and $y = 1$. The signs copied to the y-axis from the phase line diagram tell us how to draw a threaded curve, either increasing (PLUS) or decreasing (MINUS).

Labels. It is customary to use labels **sink**, **source**, **node** or the alternates **spout**, **funnel**, **node**. Additional labels are **Stable** and **Unstable**. The only stable geometry is a **sink** (**funnel**).

32 Example (Bifurcation Diagram) Verify the fish harvesting bifurcation diagram in Figure 21.

Solution: Let $f(y) = y(4 - y) - k$, where k is a parameter that controls the harvesting rate per annum. A phase line diagram is made for each relevant value of k, by applying Theorem 3 to the equilibrium points. First, the *equilibria* are computed, that is, the roots of $f(y) = 0$:

$$y^2 - 4y + k = 0 \qquad \text{Standard quadratic form of } f(y) = 0.$$

$$y = \frac{4 \pm \sqrt{4^2 - 4k}}{2} \qquad \text{Apply the quadratic formula.}$$

$$= 2 + \sqrt{4 - k}, \ \ 2 - \sqrt{4 - k} \qquad \text{Evaluate. Real roots exist only for } 4-k \geq 0.$$

In preparation to apply Theorem 3, the derivative f' is calculated and then evaluated at the equilibria:

$$f'(y) = (4y - y^2 - k)' \qquad \text{Computing } f' \text{ from } f(y) = (4 - y)y - k.$$

$$= 4 - 2y \qquad \text{Derivative found.}$$

$$f'(2 + \sqrt{4 - k}) = -2\sqrt{4 - k} \qquad \text{Negative means a } sink, \text{ by Theorem 3.}$$

$$f'(2 - \sqrt{4 - k}) = 2\sqrt{4 - k} \qquad \text{Positive means a } source, \text{ by Theorem 3.}$$

A typical phase line diagram then looks like Figure 15, page 51. In the ky-plane, sources go through the curve $y = 2 - \sqrt{4 - k}$ and sinks go through the curve $y = 2 + \sqrt{4 - k}$. This justifies the bifurcation diagram in Figure 21, and also Figure 22, except for the common point of the two curves at $k = 4$, $y = 2$.

At this common point, the differential equation is $y' = -(y-2)^2$. This equation is studied in Example 29, page 57; a change of variable $Y = 2 - y$ shows that the equilibrium is a node.

Proofs and Details

Stability Test Proof: Let f and f' be continuous. It will be justified that the equation $y' = f(y)$ has a *stable* equilibrium at $y = y_0$, provided $f(y_0) = 0$ and $f'(y_0) < 0$. The *unstable* case is left for the exercises.

We show that f changes sign at $y = y_0$ from positive to negative, as follows, hence the hypotheses of **(a)** hold. Continuity of f' and the inequality $f'(y_0) < 0$ imply $f'(y) < 0$ on some small interval $|y - y_0| \leq H$. Therefore, $f(y) > 0 = f(y_0)$ for $y < y_0$ and $f(y) < 0 = f(y_0)$ for $y > y_0$. This justifies that the hypotheses of **(a)** apply. We complete the proof using only these hypotheses.

Global existence. It has to be established that some constant $H > 0$ exists, such that $|y(0) - y_0| < H$ implies $y(x)$ exists for $x \geq 0$ and $\lim_{x \to \infty} y(x) = y_0$. To define $H > 0$, assume $f(y_0) = 0$ and the change of sign condition $f(y) > 0$ for $y_0 - H \leq y < y_0$, $f(y) < 0$ for $y_0 < y \leq y_0 + H$.

Assume that $y(x)$ exists as a solution to $y' = f(y)$ on $0 \le x \le h$. It will be established that $|y(0) - y_0| < H$ implies $y(x)$ is monotonic and satisfies $|y(x) - y_0| \le Hh$ for $0 \le x \le h$.

The constant solution y_0 cannot cross any other solution, therefore a solution with $y(0) > y_0$ satisfies $y(x) > y_0$ for all x. Similarly, $y(0) < y_0$ implies $y(x) < y_0$ for all x.

The equation $y' = f(y)$ dictates the sign of y', as long as $0 < |y(x) - y_0| \le H$. Then $y(x)$ is either decreasing ($y' < 0$) or increasing ($y' > 0$) towards y_0 on $0 \le x \le h$, hence $|y(x) - y_0| \le H$ holds as long as the monotonicity holds. Because the signs endure on $0 < x \le h$, then $|y(x) - y_0| \le H$ holds on $0 \le x \le h$.

Extension to $0 \le x < \infty$. Differential equations extension theory applied to $y' = f(y)$ says that a solution satisfying on its domain $|y(x)| \le |y_0| + H$ may be extended to $x \ge 0$. This dispenses with the technical difficulty of showing that the domain of $y(x)$ is $x \ge 0$. Unfortunately, details of proof for extension results require more mathematical background than is assumed for this text; see Birkhoff-Rota [**?**], which justifies the extension from the Picard theorem.

Limit at $x = \infty$. It remains to show that $\lim_{x \to \infty} y(x) = y_1$ and $y_1 = y_0$. The limit equality follows because y is monotonic. The proof concludes when $y_1 = y_0$ is established.

Already, $y = y_0$ is the only root of $f(y) = 0$ in $|y - y_0| \le H$. This follows from the change of sign condition in **(a)**. It suffices to show that $f(y_1) = 0$, because then $y_1 = y_0$ by uniqueness.

To verify $f(y_1) = 0$, apply the fundamental theorem of calculus with $y'(x)$ replaced by $f(y(x))$ to obtain the identity

$$y(n+1) - y(n) = \int_n^{n+1} f(y(x))dx.$$

The integral on the right limits as $n \to \infty$ to the constant $f(y_1)$, by the integral mean value theorem of calculus, because the integrand has limit $f(y_1)$ at $x = \infty$. On the left side, the difference $y(n+1) - y(n)$ limits to $y_1 - y_1 = 0$. Therefore, $0 = f(y_1)$.

The additional test stated in the theorem is the observation that internal to the proof we used only the change of sign of f at $y = y_0$, which was deduced from the sign of the derivative $f'(y_0)$. If $f'(y_0) = 0$, but the change of sign occurs, then the details of proof still apply. The proof is complete.

Exercises 1.5

Stability-Instability Test. Find all equilibria for the given differential equation and then apply Theorem 3, page 54, to obtain a classification of each equilibrium as a **source**, **sink** or **node**. Do not draw a phase line diagram.

1. $P' = (2 - P)P$

2. $P' = (1 - P)(P - 1)$

3. $y' = y(2 - 3y)$

4. $y' = y(1 - 5y)$

5. $A' = A(A - 1)(A - 2)$

6. $A' = (A - 1)(A - 2)^2$

7. $w' = \dfrac{w(1 - w)}{1 + w^2}$

8. $w' = \dfrac{w(2 - w)}{1 + w^4}$

9. $v' = \dfrac{v(1+v)}{4+v^2}$

10. $v' = \dfrac{(1-v)(1+v)}{2+v^2}$

Phase Line Diagram. Draw a phase line diagram, with detail similar to Figure 20.

11. $y' = y(2-y)$

12. $y' = (y+1)(1-y)$

13. $y' = (y-1)(y-2)$

14. $y' = (y-2)(y+3)$

15. $y' = y(y-2)(y-1)$

16. $y' = y(2-y)(y-1)$

17. $y' = \dfrac{(y-2)(y-1)}{1+y^2}$

18. $y' = \dfrac{(2-y)(y-1)}{1+y^2}$

19. $y' = \dfrac{(y-2)^2(y-1)}{1+y^2}$

20. $y' = \dfrac{(y-2)(y-1)^2}{1+y^2}$

Phase Portrait. Draw a phase portrait of threaded curves, using the phase line diagram constructed in the previous ten exercises.

21. $y' = y(2-y)$

22. $y' = (y+1)(1-y)$

23. $y' = (y-1)(y-2)$

24. $y' = (y-2)(y+3)$

25. $y' = y(y-2)(y-1)$

26. $y' = y(2-y)(y-1)$

27. $y' = \dfrac{(y-2)(y-1)}{1+y^2}$

28. $y' = \dfrac{(2-y)(y-1)}{1+y^2}$

29. $y' = \dfrac{(y-2)^2(y-1)}{1+y^2}$

30. $y' = \dfrac{(y-2)(y-1)^2}{1+y^2}$

Bifurcation Diagram. Draw a stack of phase line diagrams and construct from it a succinct bifurcation diagram with abscissa k and ordinate $y(0)$. Don't justify details at a bifurcation point.

31. $y' = (2-y)y - k$

32. $y' = (3-y)y - k$

33. $y' = (2-y)(y-1) - k$

34. $y' = (3-y)(y-2) - k$

35. $y' = y(2-y)(y-1) - k$

36. $y' = y(2-y)(y-2) - k$

37. $y' = y(y-1)^2 - k$

38. $y' = y^2(y-1) - k$

39. $y' = y(0.5 - 0.001y) - k$

40. $y' = y(0.4 - 0.045y) - k$

Details and Proofs. Supply details for the following statements.

41. **(Stability Test)** Verify **(b)** of Theorem 3, page 54, by altering the proof given in the text for **(a)**.

42. **(Stability Test)** Verify **(b)** of Theorem 3, page 54, by means of the change of variable $x \to -x$.

43. **(Autonomous Equations)** Let $y' = f(y)$ have solution $y(x)$ on $a < x < b$. Then for any c, $a < c < b$, the function $z(x) = y(x+c)$ is a solution of $z' = f(z)$.

44. **(Autonomous Equations)** The method of isoclines can be applied to an autonomous equation $y' = f(y)$ by choosing equally spaced horizontal lines $y = c_i$, $i = 1, \ldots, k$. Along each horizontal line $y = c_i$ the slope is a constant $M_i = f(c_i)$, and this determines the set of invented slopes $\{M_i\}_{i=1}^k$ for the method of isoclines.

1.6 Computing and Existence

The initial value problem

(1) $$y' = f(x, y), \quad y(x_0) = y_0$$

is studied here from a computational viewpoint. Answered are some basic questions about practical and theoretical computation of solutions:

- Why do numerical methods fail in problem (1)?
- What hypotheses for (1) make it possible to use numerical methods?
- When does (1) have a symbolic solution, that is, a solution described by an xy-equation?

Three Key Examples

The range of unusual behavior of solutions to $y' = f(x, y)$, $y(x_0) = y_0$ can be illustrated by three examples.

(A) $y' = 3(y-1)^{2/3}$, $\qquad$ The right side $f(x,y)$ is continuous. It
$y(0) = 1$. $\qquad$ has two solutions $y = 1 + x^3$ and $y = 1$.

(B) $y' = \dfrac{2y}{x-1}$, $\qquad$ The right side $f(x,y)$ is discontinuous. It
$y(0) = 1$. $\qquad$ has infinitely many piecewise-defined solutions
$$y = \begin{cases} (x-1)^2 & x < 1, \\ c(x-1)^2 & x \geq 1. \end{cases}$$

(C) $y' = 1 + y^2$, $\qquad$ The right side $f(x,y)$ is differentiable.
$y(0) = 0$. $\qquad$ It has unique solution $y = \tan(x)$, but
$y(x_0) = \infty$ at finite time $x_0 = \pi/2$.

Numerical method failure can be caused by multiple solutions to problem (1), e.g., examples (A) and (B), because a numerical method is going to compute just *one* answer; see Example 33, page 68. Multiple solutions are often signaled by discontinuity of either f or its partial derivative f_y. In (A), the right side $3(y-1)^{2/3}$ has an infinite partial at $y = 1$, while in (B), the right side $2y/(x-1)$ is infinite at $x = 1$.

Simple jump discontinuities, or **switches**, appear in modern applications of differential equations. Therefore, it is important to allow $f(x, y)$ to be discontinuous, in a limited way, but multiple solutions must be avoided, e.g., example (B). An important success story in electrical engineering is circuit theory with periodic and piecewise-defined inputs. See Example 34, page 69.

Discontinuities of f or f_y in problem (1) should raise questions about the applicability of numerical methods. Exactly why there is not a precise and foolproof test to predict failure of a numerical method remains to be explained.

Theoretical solutions exist for problem (1), if $f(x, y)$ is *continuous*. See Peano's theorem, page 67. This solution may blow up in a finite interval, e.g., $y = \tan(x)$ in example (C). See Example 35, page 70.

No symbolic closed-form solution formula exists as a result of the basic theory. In part, this dilemma is due to the possibility of multiple solutions, if f is only continuous, e.g., example (A). Picard's iteration provides assumptions to give a symbolic solution formula. However, Picard's formula is currently impractical for applied mathematics. Additional general assumptions do not seem to help. *There is in general no symbolic solution formula available for use in applied mathematics.*

Exactly one theoretical solution exists in problem (1), provided $f(x, y)$ and $f_y(x, y)$ are continuous; see the Picard–Lindelöf theorem, page 67. The situation with numerical methods improves dramatically: the most popular methods work on a computer.

Why Not "Put it on the computer?"

Typically, scientists and engineers rely upon computer algebra systems and numerical laboratories, e.g., `maple`, `mathematica` and `matlab`.

Computerization of differential equations constantly improves, with the advent of computer algebra systems and ever-improving numerical methods. Indeed, neither an advanced degree in mathematics nor a wizard's hat is required to query these systems for a closed-form solution formula. Many cases are checked systematically in a few seconds.

Fail-safe mechanisms usually do not exist for applying modern software to the initial value problem

$$\frac{dy}{dx} = f(x, y(x)), \quad y(x_0) = y_0.$$

For the initial value problem $y' = 3(y - 1)^{2/3}$, $y(0) = 1$, the computer algebra system `maple` reports the *solution* $y = 1 + x^3$. But the obvious equilibrium solution $y = 1$ is unreported. The `maple` numeric solver silently accepts the same problem and solves to obtain the solution $y = 1$. To experience this, execute the `maple` code below.

```
de:=diff(y(x),x)=3*(y(x)-1)^(2/3):  ic:=y(0)=1:
dsolve({de,ic},y(x));                    # Symbolic sol
p:=dsolve({de,ic},y(x),numeric); p(1); # Numerical sol
```

There was an improvement in the report in version 10. In later versions, $y(x) = 1$ was reported for both. The inference for the maple user is that

there is a unique solution, but the model has multiple solutions, making both reports incorrect. Computer algebra system `Maxima` has similar difficulties.

Numerical instability is typically not reported by computer software. To understand the difficulty, consider the differential equation

$$y' = y - 2e^{-x}, \quad y(0) = 1.$$

The symbolic solution is $y = e^{-x}$. Attempts to solve the equation numerically will inevitably compute the nearby solutions $y = ce^x + e^{-x}$, where c is small. As x grows, the numerical solution grows like e^x, and $|y| \to \infty$. For example, `maple` computes $y(30) \approx -72557$, but $e^{-30} \approx 0.94 \times 10^{-13}$. In reality, the solution $y = e^{-x}$ cannot be computed. The `maple` code:

```
de:=diff(y(x),x)=y(x)-2*exp(-x):  ic:=y(0)=1:
p:=dsolve({de,ic},y(x),numeric):  p(30);
```

Mathematical model formulation, prior to using a computer, seems to be an essential skill which does not come in the colorfully decorated package from the software vendor. It is this creative skill that separates the practicing scientist from the person on the street who has enough money to buy a computer program.

Closed-Form Existence-Uniqueness Theory

The **closed-form existence-uniqueness theory** describes models

$$(2) \qquad\qquad y' = f(x, y), \quad y(x_0) = y_0$$

for which a closed-form solution is known, as an equation of some sort. The objective of the theory for first order differential equations is to obtain existence and uniqueness by exhibiting a solution formula. The mathematical literature which documents these models is too vast to catalog in a textbook. We discuss only the most popular models.

`Dsolve` **Engine in** `Maple`. This computer algebra system has an implementation for some specialized equations within the closed-form theory. Below are some of the equation types examined by `maple` for solving a differential equation using classification methods. Not everything tried by `maple` is listed, e.g., Lie symmetry methods, which are beyond the scope of this text.

Equation Type	Differential Equation
Quadrature	$y' = F(x)$
Linear	$y' + p(x)y = r(x)$
Separable	$y' = f(x)g(y)$

Abel	$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$
Bernoulli	$y' + p(x)y = r(x)y^n$
Clairaut	$y = xy' + g(y')$
d'Alembert	$y = xf(y') + g(y')$
Chini	$y' = f(x)y^n - g(x)y + h(x)$
Homogeneous	$y' = f(y/x), \quad y' = y/x + g(x)f(y/x),$ $y' = (y/x)F(y/x^\alpha),$ $y' = F((a_1x + a_2y + a_3)/(a_4x + a_5y + a_6))$
Rational	$y' = P_1(x,y)/P_2(x,y)$
Ricatti	$y' = f(x)y^2 + g(x)y + h(x)$

Not every equation can be solved as written — restrictions are made on the parameters. Omitted from the above list are **power series methods** and differential equations with **piecewise-defined coefficients**. They are part of the closed-form theory but use specialized representations of solutions.

Special Equation Preview. The program here is to catalog a short list of first order equations and their known solution formulas. The formulas establish existence of a solution to the given initial value problem. They preview what is possible; details and examples appear elsewhere in the text. The issue of uniqueness is often routinely settled, as a separate issue, by applying the Picard-Lindelöf theorem. See Theorem 5, page 67, *infra*.

First Order Linear $y' + p(x)y = r(x)$ $y(x_0) = y_0$	Let p and r be continuous on $a < x < b$. Choose any (x_0, y_0) with $a < x_0 < b$. Then $y = y_0 e^{-\int_{x_0}^x p(t)dt} + \int_{x_0}^x r(t)e^{\int_t^x p(s)ds}dt.$
First Order Separable $y' = F(x)G(y)$ $y(x_0) = y_0$	Let $F(x)$ and $G(y)$ be continuous on $a < x < b$, $c < y < d$. Assume $G(y) \neq 0$ on $c < y < d$. Choose any (x_0, y_0) with $a < x_0 < b$, $c < y_0 < d$. Then $y(x) = W^{-1}\left(\int_{x_0}^x F(t)dt\right)$ where $W(Y) = \int_{y_0}^Y du/G(u).$
First Order Analytic $y' = a(x)y + b(x)$ $y(x_0) = y_0$	Assume a and b have power series expansions in $\|x - x_0\| < h$. Then the power series $y(x) = \sum_{n=0}^\infty y_n(x - x_0)^n$ is convergent in $\|x - x_0\| < h$ and the coefficients y_n are found from the recursion $n!y_n = \left(\frac{d}{dx}\right)^{n-1}(a(x)y(x) + b(x))\Big\|_{x=x_0}.$

General Existence-Uniqueness Theory

The **general existence-uniqueness theory** describes the features of a differential equation model which make it possible to compute both theoretical and numerical solutions.

Modelers who create differential equation models generally choose the differential equation based upon the intuition gained from the *closed-form theory* and the *general theory*. Sometimes, modelers are lucky enough to refine a model to some known equation with closed-form solution. Other times, they are glad for just a numerical solution to the problem. In any case, they want a model that is *tested* and *proven* in applications.

General Existence Theory in Applications. For scientists and engineers, the results can be recorded as the following statement:

> In applications it is usually enough to require $f(x, y)$ and $f_y(x, y)$ to be continuous. Then the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ is a well-tested model to which classical numerical methods apply.

The general theory results to be stated are due to **Peano** and to **Picard-Lindelöf**. The techniques of proof require advanced calculus, perhaps graduate real-variable theory as well.

Theorem 4 (Peano)
Let $f(x, y)$ be continuous in a domain D of the xy-plane and let (x_0, y_0) belong to the interior of D. Then there is a small $h > 0$ and a function $y(x)$ continuously differentiable on $|x - x_0| < h$ such that $(x, y(x))$ remains in D for $|x - x_0| < h$ and $y(x)$ is one solution (many more might exist) of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Definition 9 (Picard Iteration)
Define the constant function $y_0(x) = y_0$ and then define by iteration

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt.$$

The sequence $y_0(x)$, $y_1(x)$, ... is called the **sequence of Picard iterates** for $y' = f(x, y)$, $y(x_0) = y_0$. See Example 36 for computational details.

Theorem 5 (Picard-Lindelöf)
Let $f(x, y)$ and $f_y(x, y)$ be continuous in a domain D of the xy-plane. Let (x_0, y_0) belong to the interior of D. Then there is a small $h > 0$ and a

unique function $y(x)$ continuously differentiable on $|x - x_0| < h$ such that $(x, y(x))$ remains in D for $|x - x_0| < h$ and $y(x)$ solves

$$y' = f(x, y), \quad y(x_0) = y_0.$$

The equation

$$\lim_{n \to \infty} y_n(x) = y(x)$$

is satisfied for $|x - x_0| < h$ by the Picard iterates $\{y_n\}$.

The **Picard iteration** is the replacement for a closed-form solution formula in the general theory, because the Picard-Lindelöf theorem gives the uniformly convergent infinite series *solution*

$$(3) \qquad y(x) = y_0 + \sum_{n=0}^{\infty} \left(y_{n+1}(x) - y_n(x) \right).$$

Well, yes, it is a solution formula, but from examples it is seen to be currently impractical. There is as yet no known practical solution formula for the general theory.

The condition that f_y be continuous can be relaxed slightly, to include such examples as $y' = |y|$. The replacement is the following.

Definition 10 (Lipschitz Condition)
Let $M > 0$ be a constant and f a function defined in a domain D of the xy-plane. A **Lipschitz condition** is the inequality $|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$, assumed to hold for all (x, y_1) and (x, y_2) in D. The most common way to satisfy this condition is to require the partial derivative $f_y(x, y)$ to be continuous (Exercises page 73). A key example is $f(x, y) = |y|$, which is non-differentiable at $y = 0$, but satisfies a Lipschitz condition with $M = 1$.

Theorem 6 (Extended Picard-Lindelöf)
Let $f(x, y)$ be continuous and satisfy a Lipschitz condition in a domain D of the xy-plane. Let (x_0, y_0) belong to the interior of D. Then there is a small $h > 0$ and a *unique* function $y(x)$ continuously differentiable on $|x - x_0| < h$ such that $(x, y(x))$ remains in D for $|x - x_0| < h$ and $y(x)$ solves

$$y' = f(x, y), \quad y(x_0) = y_0.$$

The equation

$$\lim_{n \to \infty} y_n(x) = y(x)$$

is satisfied for $|x - x_0| < h$ by the Picard iterates $\{y_n\}$.

33 Example (Numerical Method Failure) A project models $y = 1 + x^3$ as the solution of the problem $y' = 3(y - 1)^{2/3}$, $y(0) = 1$. Computer work gives the solution as $y = 1$. Is it bad computer work or a bad model?

Solution: It is a bad model, explanation to follow.

Solution verification. One solution of the initial value problem is given by the equilibrium solution $y = 1$. Another is $y = 1 + x^3$. Both are verified by direct substitution, using methods similar to Example 20, page 35.

Bad computer work? Technically, the computer made no mistake. Production numerical methods use only $f(x, y) = 3(y-1)^{2/3}$ and the initial condition $y(0) = 1$. They apply a fixed algorithm to find successive values of y. The algorithm is expected to be successful, that is, it will compute a list of data points which can be graphed by "connect-the-dots." The majority of numerical methods applied to this example will compute $y = 1$ for all data points.

Bad model? Yes, it is a bad model, because the model does not *define* the solution $y = 1 + x^3$. The lesson here is that knowing a solution to an equation does not guarantee a numerical laboratory will be able to compute it.

Detecting a bad model. The right side $f(x, y)$ of the differential equation is continuous, but not continuously differentiable, therefore Picard's theorem does not apply, although Peano's theorem says a solution exists. Peano's theorem allows multiple solutions, but Picard's theorem does not. Sometimes, the only signal for non-uniqueness is the failure of application of Picard's theorem.

Physically significant models can have multiple solutions. A key example is the tank draining equation of E. Torricelli (1608-1647). A simple instance is $y' = -2\sqrt{|y|}$, in which $y(x)$ is the water height in some cylindrical tank at time x. An initial condition $y(0) = 0$ does not determine a unique solution, because the tank could have drained at some time in the past (no water in the tank means the water height $y = 0$). For instance, if the tank drained at time $x = -1$, then a piecewise defined solution is $y(x) = (x + 1)^2$ for $-\infty < x < -1$ and $y(x) = 0$ for $x \geq -1$. Most numerical methods applied to $y = -2\sqrt{|y|}$, $y(0) = 0$, compute $y(x) = 0$ for all x, which illustrates the inability of computer software to detect multiple solution errors.

34 Example (Switches) The problem $y' = f(x, y)$, $y(0) = y_0$ with

$$f(x, y) = \begin{cases} 0 & 0 \leq x \leq 1, \\ 1 & 1 < x < \infty, \end{cases}$$

has a piecewise-defined discontinuous right side $f(x, y)$. Solve the initial value problem for $y(x)$.

Solution: The solution $y(x)$ is found by dissection of the problem according to the two intervals $0 \leq x \leq 1$ and $1 < x < \infty$ into the two differential equations $y' = 0$ and $y' = 1$. By the fundamental theorem of calculus, the answers are $y = y_0$ and $y = x + y_1$, where y_1 is to be determined. At the common point $x = 1$, the two solutions should agree (we ask y to be continuous at $x = 1$), therefore $y_0 = 1 + y_1$, giving the final solution

$$y(x) = \begin{cases} y_0 & \text{on } 0 \leq x \leq 1, \\ x - 1 + y_0 & \text{on } 1 < x < \infty. \end{cases}$$

The function $y(x)$ is continuous, but y' is discontinuous at $x = 1$. The differential equation $y' = f(x, y)$ and initial condition $y(0) = y_0$ are formally satisfied.

Is there another continuous solution? No, because the method applied here assumed that $y(x)$ worked in the differential equation, and if it did, then it had to agree with $y = y_0$ on $0 \le x < 1$ and $y = x - 1 + y_0$ on $1 < x < \infty$, by the fundamental theorem of calculus.

Technically adept readers will find a flaw in the solution presented here, because of the treatment of the point $x = 1$, where y' does not exist. The flaw vanishes if we agree to verify the differential equation except at finitely many x-values where y' is undefined.

35 Example (Finite Blowup) Verify that $dx/dt = 2 + 2x^2$, $x(0) = 0$ has the unique solution $x(t) = \tan(2t)$, which approaches infinity in finite time $t = \pi/4$.

Solution: The function $x(t) = \tan(2t)$ works in the initial condition $x(0) = 0$, because $\tan(0) = \sin(0)/\cos(0) = 0$. The well-known trigonometric identity $\sec^2(2t) = 1 + \tan^2(2t)$ and the differentiation identity $x'(t) = 2\sec^2(2t)$ shows that the differential equation $x' = 2 + 2x^2$ is satisfied. The equation $x(\pi/4) = \infty$ is verified from $\tan(2t) = \sin(2t)/\cos(2t)$. Uniqueness follows from Picard's theorem, because $f(t, x) = 2 + 2x^2$ and $f_x(t, x) = 4x$ are continuous everywhere.

36 Example (Picard Iterates) Compute the Picard iterates $y_0, \ldots, y_3$ for the initial value problem $y' = f(x, y)$, $y(0) = 1$, given $f(x, y) = x - y$.

Solution: The answers are

$$y_0(x) = 1,$$
$$y_1(x) = 1 - x + x^2/2,$$
$$y_2(x) = 1 - x + x^2 - x^3/6,$$
$$y_3(x) = 1 - x + x^2 - x^3/3 + x^4/24.$$

The details for all computations are similar. A sample computation appears below.

$y_0 = 1$	This follows from $y(0) = 1$.
$y_1 = 1 + \int_0^x f(t, y_0(t))dt$	Use the formula with $n = 1$.
$\quad = 1 + \int_0^x (t - 1)dt$	Substitute $y_0(x) = 1$, $f(x, y) = x - y$.
$\quad = 1 - x + x^2/2$	Evaluate the integral.

The exact answer is $y = x - 1 + 2e^{-x}$. A Taylor series expansion of this function motivates why the Picard iterates converge to $y(x)$. See the exercises for details, page 72.

The `maple` code which does the various computations appears below. The code involving `dsolve` is used to compute the exact solution and the series solution.

```
y0:=x->1:f:=(x,y)->x-y:
y1:=x->1+int(f(t,y0(t)),t=0..x):
y2:=x->1+int(f(t,y1(t)),t=0..x):
y3:=x->1+int(f(t,y2(t)),t=0..x):
y0(x),y1(x),y2(x),y3(x);
de:=diff(y(x),x)=f(x,y(x)): ic:=y(0)=1:
dsolve({de,ic},y(x)); dsolve({de,ic},y(x),series);
```

Exercises 1.6

Multiple Solution Example. Define $f(x, y) = 3(y - 1)^{2/3}$. Consider $y' = f(x, y)$, $y(0) = 1$.

1. Do an answer check for $y(x) = 1$. Do a second answer check for $y(x) = 1 + x^3$.

2. Let $y(x) = 1$ on $0 \le x \le 1$ and $y(x) = 1 + (x - 1)^3$ for $x \ge 1$. Do an answer check for $y(x)$.

3. Does $f_y(x, y)$ exist for all (x, y)?

4. Verify that Picard's theorem does not apply to $y' = f(x, y)$, $y(0) = 1$, due to discontinuity of f_y.

5. Verify that Picard's theorem applies to $y' = f(x, y)$, $y(0) = 2$.

6. Let $y(x) = 1 + (x + 1)^3$. Do an answer check for $y' = f(x, y)$, $y(0) = 2$. Does another solution exist?

Discontinuous Equation Example. Consider $y' = \dfrac{2y}{x - 1}$, $y(0) = 1$. Define $y_1(x) = (x - 1)^2$ and $y_2(x) = c(x - 1)^2$. Define $y(x) = y_1(x)$ on $-\infty < x < 1$ and $y(x) = y_2(x)$ on $1 < x < \infty$. Define $y(1) = 0$.

7. Do an answer check for $y_1(x)$ on $-\infty < x < 1$. Do an answer check for $y_2(x)$ on $1 < x < \infty$. Skip condition $y(0) = 1$.

8. Justify one-sided limits $y(1+) = y(1-) = 0$. The functions y_1 and y_2 join continuously at $x = 1$ with common value zero and the formula

for $y(x)$ gives one continuous formal solution for each value of c (∞-many solutions).

9. (a) For which values of c does $y'(1)$ exist? (b) For which values of c is $y(x)$ continuously differentiable?

10. Find all values of c such that $y(x)$ is a continuously differentiable function that satisfies the differential equation and the initial condition.

Finite Blowup Example. Consider $y' = 1 + y^2$, $y(0) = 0$. Let $y(x) = \tan x$.

11. Do an answer check for $y(x)$.

12. Find the partial derivative f_y for $f(x, y) = 1 + y^2$. Justify that f and f_y are everywhere continuous.

13. Justify that Picard's theorem applies, hence $y(x)$ is the only possible solution to the initial value problem.

14. Justify for $a = -\pi/2$ and $b = \pi/2$ that $y(a+) = -\infty$, $y(b-) = \infty$. Hence $y(x)$ blows up for finite values of x.

Numerical Instability Example. Let $f(x, y) = y - 2e^{-x}$.

15. Do an answer check for $y(x) = e^{-x}$ as a solution of the initial value problem $y' = f(x, y)$, $y(0) = 1$.

16. Do an answer check for $y(x) = ce^x + e^{-x}$ as a solution of $y' = f(x, y)$.

Multiple Solutions. Consider the initial value problem $y' = 5(y - 2)^{4/5}$, $y(0) = 2$.

17. Do an answer check for $y(x) = 2$. Do a second answer check for $y(x) = 2 + x^5$.

18. Verify that the hypotheses of Picard's theorem fail to apply.

19. Find a formula which displays infinitely many solutions to $y' = f(x, y)$, $y(0) = 2$.

20. Verify that the hypotheses of Peano's theorem apply.

Discontinuous Equation. Consider $y' = \dfrac{y}{x - 1}$, $y(0) = 1$. Define $y(x)$ piecewise by $y(x) = -(x-1)$ on $-\infty < x < 1$ and $y(x) = c(x - 1)$ on $1 < x < \infty$. Leave $y(1)$ undefined.

21. Do an answer check for $y(x)$. The initial condition $y(0) = 1$ applies only to the domain $-\infty < x < 1$.

22. Justify one-sided limits $y(1+) = y(1-) = 0$. The piecewise definitions of $y(x)$ join continuously at $x = 1$ with common value zero and the formula for $y(x)$ gives one continuous formal solution for each value of c (∞-many solutions).

23. (a) For which values of c does $y'(1)$ exist? (b) For which values of c is $y(x)$ continuously differentiable?

24. Find all values of c such that $y(x)$ is a continuously differentiable function that satisfies the differential equation and the initial condition.

Picard Iteration. Find the Picard iterates y_0, y_1, y_2, y_3.

25. $y' = y + 1$, $y(0) = 2$

26. $y' = y + 1$, $y(0) = 2$

27. $y' = y + 1$, $y(0) = 0$

28. $y' = 2y + 1$, $y(0) = 0$

29. $y' = y^2$, $y(0) = 1$

30. $y' = y^2$, $y(0) = 2$

31. $y' = y^2 + 1$, $y(0) = 0$

32. $y' = 4y^2 + 4$, $y(0) = 0$

33. $y' = y + x$, $y(0) = 0$

34. $y' = y + 2x$, $y(0) = 0$

Picard Iteration and Taylor Series. Find the Taylor polynomial $P_n(x) = y(0) + y'(0)x + \cdots + y^{(n)}(0)x^n/n!$ and compare with the Picard iterates. Use a computer algebra system, if possible.

35. $y' = y$, $y(0) = 1$, $n = 4$, $y(x) = e^x$

36. $y' = 2y$, $y(0) = 1$, $n = 4$, $y(x) = e^{2x}$

37. $y' = x - y$, $y(0) = 1$, $n = 4$, $y(x) = -1 + x + 2e^{-x}$

38. $y' = 2x - y$, $y(0) = 1$, $n = 4$, $y(x) = -2 + 2x + 3e^{-x}$

Numerical Instability. Use a computer algebra system or numerical laboratory. Let $f(x, y) = y - 2e^{-x}$.

39. Solve $y' = f(x, y)$, $y(0) = 1$ numerically for $y(30)$.

40. Solve $y' = f(x, y)$, $y(0) = 1 + 0.0000001$ numerically for $y(30)$.

Closed–Form Existence. Solve these initial value problems using a computer algebra system.

41. $y' = y$, $y(0) = 1$

42. $y' = 2y$, $y(0) = 2$

43. $y' = 2y + 1$, $y(0) = 1$

44. $y' = 3y + 2$, $y(0) = 1$

45. $y' = y(y - 1)$, $y(0) = 2$

46. $y' = y(1 - y)$, $y(0) = 2$

47. $y' = (y - 1)(y - 2)$, $y(0) = 3$

48. $y' = (y - 2)(y - 3)$, $y(0) = 1$

49. $y' = -10(1 - y)$, $y(0) = 0$

50. $y' = -10(2 - 3y)$, $y(0) = 0$

Lipschitz Condition. Justify the following results.

51. The function $f(x, y) = x - 10(2 - 3y)$ satisfies a Lipschitz condition on the whole plane.

52. The function $f(x, y) = ax + by + c$ satisfies a Lipschitz condition on the whole plane.

53. The function $f(x, y) = xy(1 - y)$ satisfies a Lipschitz condition on $D = \{(x, y) : |x| \leq 1, \quad |y| \leq 1\}$.

54. The function $f(x, y) = x^2 y(a - by)$ satisfies a Lipschitz condition on $D = \{(x, y) : x^2 + y^2 \leq R^2\}$.

55. If f_y is continuous on D and the line segment from y_1 to y_2 is in D, then $f(x, y_1) - f(x, y_2) = \int_{y_1}^{y_2} f_y(x, u) du$.

56. If f and f_y are continuous on a disk D, then f is Lipschitz with $M = \max_D\{|f_y(x, u)|\}$.