Cayley-Hamilton Theorem

- Characteristic Equation
- Cayley-Hamilton Theorem
- An Example

Characteristic Equation

Definition 1 (Characteristic Equation)
Given a square matrix $\boldsymbol{A}$, the **characteristic equation** of $\boldsymbol{A}$ is the polynomial equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = 0.$$

The determinant $|\boldsymbol{A} - r\boldsymbol{I}|$ is formed by subtracting $\boldsymbol{r}$ from the diagonal of $\boldsymbol{A}$.
The polynomial $p(r) = |\boldsymbol{A} - r\boldsymbol{I}|$ is called the **characteristic polynomial**.

- If $\boldsymbol{A}$ is 2×2, then $p(r)$ is a quadratic.
- If $\boldsymbol{A}$ is 3×3, then $p(r)$ is a cubic.
- The determinant is expanded by the cofactor rule, in order to preserve factorizations.

Characteristic Equation Examples

Create $\det(A - rI)$ by subtracting r from the diagonal of A.

Evaluate by the cofactor rule.

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2 - r & 3 \\ 0 & 4 - r \end{vmatrix} = (2 - r)(4 - r)$$

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2 - r & 3 & 4 \\ 0 & 5 - r & 6 \\ 0 & 0 & 7 - r \end{vmatrix} = (2-r)(5-r)(7-r)$$

Cayley-Hamilton

Theorem 1 (Cayley-Hamilton)
A square matrix $\boldsymbol{A}$ satisfies its own characteristic equation.

If $\boldsymbol{p(r) = (-r)^n + a_{n-1}(-r)^{n-1} + \cdots a_0}$, then the result is the equation

$$\boldsymbol{(-A)^n + a_{n-1}(-A)^{n-1} + \cdots + a_1(-A) + a_0 I = 0,}$$

where $\boldsymbol{I}$ is the $\boldsymbol{n \times n}$ identity matrix and $\boldsymbol{0}$ is the $\boldsymbol{n \times n}$ zero matrix.

The 2 × 2 Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the cofactor rule implies

$$p(r) = \begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = r^2 - (a+d)r + ad - bc.$$

Define

$$\begin{aligned} a_1 &= a + d = \mathbf{trace}(A), \\ a_0 &= ad - bc = \det(A). \end{aligned}$$

Then $p(r) = r^2 + a_1(-r) + a_0$. The Cayley-Hamilton theorem says

$$A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ or}$$

$$A^2 - (a+d)A + (ad-bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Cayley-Hamilton Example

Assume

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$$

Then

$$p(r) = \begin{vmatrix} 2-r & 3 & 4 \\ 0 & 5-r & 6 \\ 0 & 0 & 7-r \end{vmatrix} = (2-r)(5-r)(7-r)$$

and the Cayley-Hamilton Theorem says that

$$(2I - A)(5I - A)(7I - A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$