Solution Set Basis for Linear Differential Equations

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Linear Differential Equations

The solution set of a homogeneous constant coefficient linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$$

is known to be a vector space of functions of dimension n, consisting of special linear combinations

$$y = c_1 f_1 + \cdots + c_n f_n, \tag{1}$$

where $f_1, \ldots, f_n$ are elementary functions known as **Euler solution atoms**.

Definition of Euler Solution Atom

A **base atom** is defined to be one of

$$1, e^{ax}, \cos bx, \sin bx, e^{ax}\cos bx, e^{ax}\sin bx,$$

with real $a \neq 0$, $b > 0$.
An **Euler solution atom** equals a base atom multiplied by x^n, where $n = 0, 1, 2\ldots$ is an integer.
An atom has coefficient 1, and the zero function is not an atom.

Examples of Euler Solution Atoms

$$1,\ x,\ x^2,\ e^x,\ xe^{-x},\ x^{15}e^{2x}\cos 3x,\ \cos 3x,\ \sin 2x,\ x^2\cos 2x,\ x^6\sin \pi x,\ x^{10}e^{\pi x}\sin 0.1x$$

Functions that are not Atoms

$$x/(1+x),\ \ln|x|,\ e^{x^2},\ \sin(x+1),\ 0,\ 2x,\ \sin(1/x),\ \sqrt{x}$$

Theorems about Atoms

Theorem 1 (Independence)
Any finite list of atoms is linearly independent.

Theorem 2 (Euler)
The *real* characteristic polynomial $p(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_0$ has a factor $(r - a - ib)^{k+1}$ if and only if

$$x^k e^{ax} \cos bx, \quad x^k e^{ax} \sin bx$$

are real solutions of the differential equation (1). If $b > 0$, then both are atoms. If $b = 0$, then only the first is an atom. The theorem is applied multiple times to discover all Euler solution atoms for a given characteristic equation.

Theorem 3 (Real Solutions)
If u and v are real and $u + iv$ is a solution of equation (1), then u and v are real solutions of equation (1).

Theorem 4 (Basis)
The solution set of equation (1) has a basis of n solution atoms which are determined by Euler's theorem.

Theorem 5 (How to Apply Euler's Theorem)

Factor dividing $p(r)$	Euler Solution Atom(s)
$(r-5)$	e^{5x}
$(r+7)^2$	e^{-7x}, xe^{-7x}
$(r+7)^3$	e^{-7x}, xe^{-7x}, x^2e^{-7x}
r	e^{0x}
r^2	e^{0x} and xe^{0x}
r^3	1, x and x^2 $[e^{0x}=1]$
$(r-5i)$	$\cos 5x$ and $\sin 5x$
$(r+3i)^2$	$\cos 3x$, $x\cos 3x$, $\sin 3x$, $x\sin 3x$
$(r-2+3i)^2$	$e^{2x}\cos 3x$, $xe^{2x}\cos 3x$, $e^{2x}\sin 3x$, $xe^{2x}\sin 3x$

Example 1. Solve $y''' = 0$.

Solution: $p(r) = r^3$ implies 1 is a base atom and then 1, x, x^2 are Euler solution atoms. They are independent, hence form a basis for the 3-dimensional solution space. Then $y = c_1 + c_2 x + c_3 x^2$.

Example 2. Solve $y'' + 4y = 0$.

Solution: $p(r) = r^2 + 4$ implies base atoms $\cos 2x$ and $\sin 2x$. They are a basis for the 2-dimensional solution space with $y = c_1 \cos 2x + c_2 \sin 2x$.

Example 3. Solve $y'' + 2y' = 0$.

Solution: $p(r) = r^2 + 2r$ implies 1, e^{-2x} are base solution atoms. These independent Euler solution atoms form a basis. Then $y = c_1 + c_2 e^{-2x}$.

Example 4. Solve $y^{(4)} + 4y'' = 0$.

Solution: $p(r) = r^4 + 4r^2 = r^2(r^2 + 4)$ implies the four atoms 1, x, $\cos 2x$, $\sin 2x$ are solutions. Then $y = c_1 + c_2 x + c_3 \cos 2x + c_3 \sin 2x$.

Example 5. Solve the differential equation if $p(r) = (r^3 - r^2)(r^2 - 1)(r^2 + 4)^2$.

Solution: The distinct factors of $p(r)$ are r^2, $(r-1)^2$, $r+1$, $(r-2i)^2$, $(r+2i)^2$. Euler's theorem implies the DE has nine solution atoms 1, x, e^x, xe^x, e^{-x}, $\cos 2x$, $x \cos 2x$, $\sin 2x$, $x \sin 2x$. Then y is a linear combination of these atoms.