Phase-Amplitude Conversion, Cafe Door, Pet Door, Damping Classifications

- Phase-amplitude conversion

- Cafe door

- Pet door

  – Cafe Door Model

  – Pet Door Model

- Classifying Damped Models

Phase-amplitude conversion–I

Given a simple harmonic motion $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$, as in Figure 1, define **amplitude** A and **phase angle** α by the formulas

$$A = \sqrt{c_1^2 + c_2^2}, \ c_1 = A \cos \alpha, \ c_2 = A \sin \alpha.$$

Then the simple harmonic motion has the **phase-amplitude form**

$$(1) \qquad\qquad x(t) = A \cos(\omega t - \alpha).$$

To directly obtain (1) from trigonometry, use the trigonometric identity

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

with $a = \omega t$ and $b = \alpha$. It is known from trigonometry that $x(t)$ has **period** $2\pi/\omega$ and **phase shift** α/ω. A full period is called a **cycle** and a half-period a **semicycle**. The **frequency** $\omega/(2\pi)$ is the number of complete cycles per second, or the reciprocal of the period.
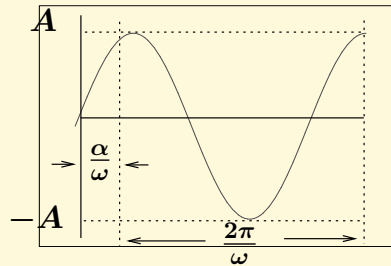
Figure 1. Simple harmonic oscillation $x(t) = A \cos(\omega t - \alpha)$, showing the period $2\pi/\omega$, the phase shift α/ω and the amplitude A.

Phase-amplitude conversion–II _____

- The **phase shift** is the amount of horizontal translation required to shift the cosine curve $\cos(\omega t - \alpha)$ so that its graph is atop $\cos(\omega t)$. To find the phase shift from $x(t)$, set the argument of the cosine term to zero, then solve for t.

- To solve for $\alpha \geq 0$ in the equations $c_1 = A \cos \alpha$, $c_2 = A \sin \alpha$, first compute numerically by calculator the radian angle $\phi = \arctan(c_2/c_1)$, which is in the range $-\pi/2$ to $\pi/2$. Quadrantal angle rules must be applied when $c_1 = 0$, because calculators return an error code for division by zero. A *common error* is to set α equal to ϕ. Not just the violation of $\alpha \geq 0$ results — the error is a fundamental one, due to trigonometric intricacies, causing us to consider the equations $c_1 = A \cos \alpha$, $c_2 = A \sin \alpha$ in order to construct the answer for α:

$$\alpha = \begin{cases} \phi & (c_1, c_2) \text{ in quadrant } I, \\ \phi + \pi & (c_1, c_2) \text{ in quadrant } II, \\ \phi + \pi & (c_1, c_2) \text{ in quadrant } III, \\ \phi + 2\pi & (c_1, c_2) \text{ in quadrant } IV. \end{cases}$$
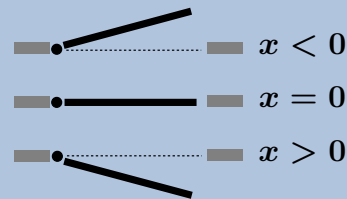
Cafe door

Restaurant waiters and waitresses are accustomed to the cafe door, which partially blocks the view of onlookers, but allows rapid trips to the kitchen – see Figure 2. The door is equipped with a spring which tries to restore the door to the equilibrium position $x = 0$, which is the plane of the door frame. There is a damper attached, to keep the number of oscillations low.
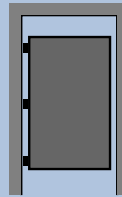
Figure 2. A cafe door on three hinges with damper in the lower hinge. The equilibrium position is the plane of the door frame.

The top view of the door, Figure 3, shows how the angle $x(t)$ from equilibrium $x = 0$ is measured from different door positions.

Figure 3. Top view of a cafe door, showing the three possible door positions.

Pet door

Designed for dogs and cats, the small door in Figure 4 allows animals to enter and exit the house freely. A pet door might have a weather seal and a security lock.


Figure 4. A pet door.
The equilibrium position is the plane of the door frame.

The pet door swings freely from hinges along the top edge. One hinge is spring–loaded with damper. Like the cafe door, the spring restores the door to the equilibrium position while the damper acts to eventually stop the oscillations. However, there is one fundamental difference: if the spring–damper system is removed, then the door continues to oscillate! The cafe door model will not describe the pet door.

Cafe Door Model ⸻

$x < 0$

$x = 0$

$x > 0$
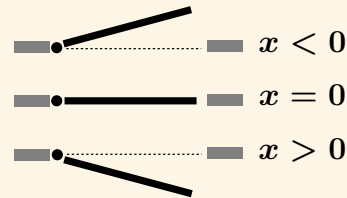
Figure 5. Top view of a cafe door, showing the three possible door positions.

Figure 5 shows that, for modeling purposes, the cafe door can be reduced to a torsional pendulum with viscous damping. This results in the **cafe door** equation

(2) $$Ix''(t) + cx'(t) + \kappa x(t) = 0.$$

The removal of the spring ($\kappa = 0$) causes the solution $x(t)$ to be monotonic, which is a reasonable fit to a springless cafe door.

Pet Door Model

Figure 6. A pet door.
The equilibrium position is the plane of the door frame.

For modeling purposes, the pet door can be compressed to a linearized swinging rod of length L (the door height). The torque $I = mL^2/3$ of the door assembly becomes important, as well as the linear restoring force kx of the spring and the viscous damping force cx' of the damper. All considered, a suitable model is the **pet door** equation

$$(3) \qquad I\,x''(t) + cx'(t) + \left(k + \frac{mgL}{2}\right) x(t) = 0.$$

Derivation of (3) is by equating to zero the algebraic sum of the forces. Removing the damper and spring ($c = k = 0$) gives a harmonic oscillator $x''(t) + \omega^2 x(t) = 0$ with $\omega^2 = 0.5mgL/I$, which establishes sanity for the modeling effort. Equation (3) is *formally* the cafe door equation with an added linearization term $0.5mgLx(t)$ obtained from $0.5mgL \sin x(t)$.

Classifying Damped Models ──────────────────────────────

Consider a differential equation

$$ay'' + by' + cy = 0$$

with constant coefficients a, b, c. It has characteristic equation $ar^2 + br + c = 0$ with roots r_1, r_2.

Classification	Defining properties
Overdamped	Distinct real roots $r_1 \neq r_2$ Positive discriminant $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ $\quad$ = exponential $\times$ monotonic function
Critically damped	Double real root $r_1 = r_2$ Zero discriminant $x = c_1 e^{r_1 t} + c_2 \, t \, e^{r_1 t}$ $\quad$ = exponential $\times$ monotonic function
Underdamped	Complex conjugate roots $\alpha \pm i\,\beta$ Negative discriminant $x = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$ $\quad$ = exponential $\times$ harmonic oscillation

Distinguishing the Three Damped Models

Consider the pet door or the cafe door, modeled by a differential equation

$$ay'' + by' + cy = 0$$

with constant coefficients a, b, c. We imagine the door closing due to adjustment of the damper screw, which affects the magnitude of coefficient b: small values of b create oscillation and large values non-oscillation.

Classification	Physical Meaning
Overdamped	The door closes slowly without oscillations.
Critically damped	The door closes without oscillations, in the least amount of time.
Underdamped	The door oscillates through the jamb position many times with decreasing amplitudes.