Undetermined Coefficients
The Trial Solution Method

- Definition of Euler Solution Atom
- Undetermined Coefficients Trial Solution Method
- Symbols
- Superposition
- The Trial Solution with Fewest Euler Atoms
- Two Rules
  - Rule I
  - Rule II
- Illustrations
- How can a variable disappear after trial solution substitution?

Definition of Euler Solution Atom

An **Euler solution atom of a linear constant-coefficient homogeneous differential equation** is briefly called an **atom**. The set of atoms is generated from base atoms and powers of x.

An **Euler base atom** is one of the terms 1, $\cos bx$, $\sin bx$, e^{ax}, $e^{ax}\cos bx$, $e^{ax}\sin bx$. An **Euler atom** equals x^n times an Euler base atom, for $n = 0, 1, 2, 3 \ldots$.

Examples.
The following are atoms: e^{2x}, $e^{e^2 x}$, $xe^{-\pi x}$, e^{0x} or 1, x, x^2, $\cos x$, $\cos \pi x$, $e^{-x}\sin 2x$, $x^6 \sin 100x$, $x^2 e^{-5x}$, $x^5 e^{-5x}\cos 5x$, 2^x [equals e^{ax} with $a = \ln 2$], any power x^n with integer $n \geq 0$.

The following are not atoms: 2, x^{-1}, $\ln|x|$, e^{x^2}, $\tan x$, $\sinh x$, $\sec x$, $\csc x$, $\sin^2 x$, $\sin(x^2)$, $e^x \cos(2x+2)$, $\cot x$, $\frac{x}{1+x}$.

Undetermined Coefficients

Step 1. Find a trial solution y by Rule I.

> **Rule I**. Assume the right side $f(x)$ of the differential equation is a linear combination of atoms. Make a list of all distinct atoms that appear in the derivatives $f(x)$, $f'(x)$, $f''(x)$, ... Multiply these k atoms by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define a **trial solution** y.
>
> > **Warning**: Rule I can **Fail**. It fails exactly when one of the atoms is a solution of the homogeneous differential equation. Apply Rule II *infra*, in case of failure of Rule I, to define trial solution y.

Step 2. Substitute trial solution y into the differential equation. The resulting equation is a competition between two linear combinations of the k atoms in the list.

Step 3. Linear independence of atoms implies matching of coefficients of atoms left and right. Write out linear algebraic equations for unknowns d_1, d_2, ..., d_k. Solve the equations.

Step 4. The trial solution y with evaluated coefficients d_1, d_2, ..., d_k becomes the particular solution y_p.

Rule I Failure

Example. The differential equation $y'' = x + e^x$ has by Rule I a trial solution $y = d_1(1) + d_2(x) + d_3(e^x)$ obtained from the list of $k = 3$ atoms 1, x, e^x. The trial solution fails to work, because upon substitution of y into the differential equation the resulting equation is

$$d_1(1)'' + d_2(x)'' + d_3(e^x)'' = 0(1) + 1(x) + 1(e^x).$$

This equation cannot be satisfied by choosing values of d_1, d_2, d_3, because it reads

$$x + (1 - d_3)e^x = 0,$$

implying that x, e^x are *dependent*, a violation of the *Independence of Atoms Theorem*.

The actual trouble is a deeper problem. The equations $(1)'' = 0$ and $(x)'' = 0$ imply that 1 and x are solutions of the homogeneous differential equation $y'' = 0$. These equations cause constants d_1, d_2 to be **completely absent** from the system of equations. The constants d_1, d_2, d_3 must be uniquely determined. A variable that is absent in a linear system is a free variable, causing non-uniqueness, and this is the root of the problem.

Symbols

The symbols c_1, c_2 are reserved for use as arbitrary constants in the general solution y_h of the homogeneous equation. For example, the homogeneous equation $y'' + y = 0$ has general solution $y = c_1 \cos x + c_2 \sin x$.

Symbols $d_1, d_2, d_3, \ldots$ are reserved for use in the trial solution y of the non-homogeneous equation. For example, the equation $y'' + y = x + e^x$ has by Rule I trial solution $y = d_1(1) + d_2(x) + d_3(e^x)$.

Abbreviations

c = constant = arbitrary constant,

d = determined constant.

Superposition

The relation $y = y_h + y_p$ suggests solving $ay'' + by' + cy = f(x)$ in two stages:

(a) Find y_h as a linear combination of atoms computed by applying Euler's theorem to factors of the characteristic polynomial $ar^2 + br + c$.

(b) Apply the **the method of undetermined coefficients** to find y_p.

Remarks

We expect to find two arbitrary constants c_1, c_2 in the solution y_h, but in contrast, **no arbitrary constants** appear in y_p.

Calling $d_1, d_2, d_3, \ldots$ *undetermined* coefficients is misleading, because in fact they are eventually *determined*.

The Trial Solution with Fewest Euler Atoms

Undetermined coefficient theory computes a **shortest possible trial solution**, a solution with **fewest Euler atoms**.

Using the fewest atoms minimizes the size of the linear algebra problem for the constants $d_1, \ldots, d_k$. A deeper property of using the fewest atoms possible is that constants $d_1, \ldots,$ d_k are *uniquely determined*.

Example. $y'' + y = x^2$

The atom list for $f(x) = x^2$ is 1, x, x^2. Rule I computes a shortest trial solution $y = d_1 + d_2 x + d_3 x^2$. The linear algebra problem is 3×3, and no smaller system of equations can be found.

The Rules for Undetermined Coefficients

Rule I. Assume the right side $f(x)$ of the differential equation is a linear combination of atoms. Make a list of all distinct atoms that appear in the derivatives $f(x)$, $f'(x)$, $f''(x)$, ... Multiply these k atoms by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define a **trial solution** y.

This rule **FAILS** if one or more of the k atoms is a solution of the homogeneous differential equation.

Rule II. If Rule I **FAILS**, then break the k atoms into groups with the same **base atom**. Cycle through the groups, replacing atoms as follows. If the first atom in the group is a solution of the homogeneous differential equation, then multiply all atoms in the group by factor x. Repeat until the first group atom is not a solution of the homogeneous differential equation. Multiply the constructed k atoms by symbols d_1, ..., d_k and add to re-define trial solution y.

Number of Euler Atoms in a Trial Solution

Theorem 1 (Number of Euler Atoms)
The number k of Euler atoms computed by **Rule I** is unchanged when applying **Rule II**. Atoms changed by **Rule II** differ only by a power of x.

An Illustration

Assume the constant-coefficient differential equation has order $\mathbf{2}$ and forcing term $f(x) = 5x^3e^{2x} + 6\sin(x) + 8e^x$. The trial solution from Rule I uses the seven (7) atoms

$$e^{2x},\ xe^{2x},\ x^2e^{2x},\ x^3e^{2x}, \cos x, \sin x, e^x.$$

Break the $\mathbf{7}$ atoms into $\mathbf{4}$ groups, each group with the same base atom.

Group	Atoms	Base Atom
1	$e^{2x},\ xe^{2x},\ x^2e^{2x},\ x^3e^{2x}$	e^{2x}
2	$\cos x$	$\cos x$
3	$\sin x$	$\sin x$
4	e^x	e^x

Example 1

Assume second order homogeneous differential equation has characteristic equation $(r-1)(r-3)=0$ and forcing term $f(x)=5x^3e^{2x}+6\sin(x)+8e^x$. Then e^{2x}, $\cos x$, $\sin x$ are **not** solutions of the homogeneous equation, but e^x **is** a solution. The solution atom e^{3x} of the homogeneous equation is not used in the trial solution construction from Rule I, which uses the seven (7) atoms

$$e^{2x},\ xe^{2x},\ x^2e^{2x},\ x^3e^{2x}, \cos x, \sin x, e^x.$$

The 4 groups are identical to the first illustration.

Rule I fails because the Group 4 atom e^x is a solution of the homogeneous equation. The other groups do not contain solutions of the homogeneous differential equation.
Rule II applies to give one new group and three unchanged groups. The trial solution y is a linear combination of the 7 atoms.

Group	Atoms
1	$e^{2x},\ xe^{2x},\ x^2e^{2x},\ x^3e^{ex}$
2	$\cos x$
3	$\sin x$
New 4	xe^x

Details. Atom xe^x is a solution of the homogeneous equation if and only if 1 is a double root of the characteristic equation; it isn't, which stops the multiplication by x in Group 4.

Example 2

Assume second order homogeneous differential equation has characteristic equation $(r-1)(r-2)=0$ and forcing term $f(x)=5x^3e^{2x}+6\sin(x)+8e^x$. Then e^x, e^{2x} are solutions of the homogeneous equation. The trial solution construction from Rule I uses the seven (7) atoms

$$e^{2x},\ xe^{2x},\ x^2e^{2x},\ x^3e^{2x}, \cos x, \sin x, e^x.$$

The 4 groups are identical to the first illustration. Then $\cos x$, $\sin x$ are **not** solutions of the homogeneous equation, but e^{2x}, e^x **are** solutions,

Rule I fails because the Group 1 atom e^{2x} is a solution of the homogeneous equation (it also fails because of Group 4). **Rule II applies** to give two new groups and two unchanged groups. The trial solution y is a linear combination of the 7 atoms.

Group	Atoms
New 1	$xe^{2x},\ x^2e^{2x},\ x^3e^{2x},\ x^4e^{ex}$
2	$\cos x$
3	$\sin x$
New 4	xe^x

Details. Atom xe^{2x} is a solution of the homogeneous equation if and only if 2 is a double root of the characteristic equation; it isn't, which stops the multiplication by x in Group 1.
Atom xe^x is a solution of the homogeneous equation if and only if 1 is a double root of the characteristic equation; it isn't, which stops the multiplication by x in Group 4.

How can a variable disappear after trial solution substitution?

Substitute trial solution $y = d_1A_1 + d_2A_2 + d_3A_3$ into DE $y'' + py' + qy = f(x) \equiv x^2$, where A_1, A_2, A_3 are atoms $1, x, x^2$ for the right side $f(x) \equiv x^2$:

$$\begin{aligned} &d_1\left.(y'' + py' + qy)\right|_{y=A_1} + \\ &d_2\left.(y'' + py' + qy)\right|_{y=A_2} + \\ &d_3\left.(y'' + py' + qy)\right|_{y=A_3} = x^2. \end{aligned}$$

The only way variable d_2 can vanish is if $\left.(y'' + py' + qy)\right|_{y=A_2} = 0$. Briefly,

The homogeneous equation

$$y'' + py' + qy = 0$$

must have the atom $(y = A_2)$ as a solution.

In general, a variable from $d_1, \ldots, d_k$ will disappear from the LHS after substitution if and only if the atom which it multiplies is a solution of the homogeneous differential equation.