The Corrected Trial Solution
in
the Method of Undetermined Coefficients

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Definition of Related Atoms

A **base atom** is one of the terms 1, $\cos bx$, $\sin bx$, e^{ax}, $e^{ax}\cos bx$, $e^{ax}\sin bx$. An **atom** equals x^n times a base atom, for $n = 0, 1, 2, 3 \ldots$.
Atoms A and B are **related** if and only if their successive derivatives A, A', A'', …, B, B', B'', … share a common atom.

Then x^3 is related to x and x^{101}, while x is unrelated to e^x, xe^x and $x\sin x$. Atoms $x\sin x$ and $x^3\cos x$ are related, but the atoms $\cos 2x$ and $\sin x$ are unrelated.

An easy way to detect related atoms:

> Atom A is related to atom B if and only if their base atoms are identical or else they would become identical by changing a sine to a cosine.

The Basic Trial Solution Method

The method is outlined here for an nth order linear differential equation.

Undetermined Coefficients Trial Solution Method

Step 1. Let $g(x) = x^n f(x)$, where n is the order of the differential equation. Extract all distinct atoms that appear in the derivatives $g(x)$, $g'(x)$, $g''(x)$, ..., then collect the distinct atoms so found into a list of k atoms. Multiply these atoms by **undetermined coefficients** d_1, ..., d_k, then add to define a **trial solution** y
.

Step 2. Substitute y into the differential equation.

Step 3. Match coefficients of atoms left and right to write out linear algebraic equations for unknowns d_1, d_2, ..., d_k. Solve the equations. Any variables not appearing are set to zero.

Step 4. The trial solution y with evaluated coefficients d_1, d_2, ..., d_k becomes the particular solution y_p.

Symbols

The symbols $\boldsymbol{c_1}$, $\boldsymbol{c_2}$ are reserved for use as arbitrary constants in the general solution $\boldsymbol{y_h}$ of the homogeneous equation.

Symbols $\boldsymbol{d_1}$, $\boldsymbol{d_2}$, $\boldsymbol{d_3}$, $\ldots$ are reserved for use in the trial solution $\boldsymbol{y}$ of the non-homogeneous equation. Abbreviations: $\boldsymbol{c} =$ constant, $\boldsymbol{d} =$ determined.

Superposition

The relation $y = y_h + y_p$ suggests solving $ay'' + by' + cy = f(x)$ in two stages:

(a) Find y_h as a linear combination of atoms computed by applying Euler's theorem to factors of the characteristic polynomial $ar^2 + br + c$.

(b) Apply the **basic trial solution method** to find y_p.

- We expect to find two arbitrary constants c_1, c_2 in the solution y_h, but in contrast, **no arbitrary constants** appear in y_p.
- Calling d_1, d_2, d_3, … *undetermined* coefficients is misleading, because in fact they are eventually *determined*.

The Trial Solution with Fewest Atoms

Undetermined coefficient theory computes a trial solution with **fewest atoms**, thereby eliminating superfluous symbols, which effects a reduction in the size of the algebra problem. In the case of the example $y'' + y = x^2$, the theory computes a trial solution $y = d_1 + d_2 x + d_3 x^2$, reducing the number of symbols from 5 to 3.

In a general equation $ay'' + by' + cy = f(x)$, the atoms in the trial solution y are the atoms that appear in $g(x) = x^2 f(x)$ plus all lower-power related atoms. Equivalently, the atoms are those extracted from the successive derivatives $g(x)$, $g'(x)$, $g''(x)$, $\ldots$. For example, if $f(x) = x^2$, then $g(x) = x^2(x^2) = x^4$ and the *list of derivatives* is x^4, $4x^3$, $12x^2$, $24x$, 24. Strip coefficients to identify *the list of related atoms* 1, x, x^2, x^3, x^4. Alternatively, begin with the atoms in $g(x)$, namely x^4, and append all lower powered related atoms. Briefly, atom x^4 causes an append of related atoms 1, x, x^2, x^3.

Two Correction Rules

The *initial* trial solution y obtained by constructing atoms from $g(x) = x^n f(x)$ is not the trial solution with fewest atoms. It is a sum of terms which can be organized into groups of related atoms, and it is known that each group contains n superfluous terms. The correction rules describe how to remove the superfluous terms, which produces the desired corrected trial solution with **fewest possible atoms**.

Correction Rule I

If some variable d_p is missing after substitution **Step 2**, then the system of linear equations for $d_1, \ldots, d_k$ fails to have a unique solution. In the language of linear algebra, a missing variable d_p in the system of linear equations is a *free variable*, which implies the linear system in the unknowns $d_1, \ldots, d_k$ has, among the *three possibilities*, infinitely many solutions.

A symbol d_p appearing in a trial solution will be missing in **Step 2** if and only if it multiplies an atom $A(x)$ that is a solution of the homogeneous equation. Because d_p will be a free variable [any missing variable is a free variable], to which we will assign value zero in **Step 3**, the term $d_p A(x)$ can be removed from the trial solution. We can do this in advance, to **decrease the number of symbols** in the trial solution.

> **Rule I**. Remove all terms $d_p A(x)$ in the trial solution of **Step 1** for which atom $A(x)$ is a solution of the homogeneous differential equation.

Correction Rule II

The trial solution always contains superfluous atoms, introduced by using $x^n f(x)$ to construct the trial solution instead of $f(x)$. For example, the equation $y'' + y = x^2$ would have trial solution $y = d_1 + d_2 x + d_3 x^2 + d_4 x^3 + d_5 x^4$, with atoms x^3 and x^4 superfluous, because $y_p = x^2 - 2$. We could have replaced the 5-term trial solution by 3-termed trial solution $y = d_1 + d_2 x + d_3 x^2$. There is a rule for how to remove superfluous terms, which combines easily with Rule I:

Rule II. Terms removed from Rule I appear in groups of related atoms

$$B(x), \quad xB(x), \quad \ldots, \quad x^m B(x),$$

where $B(x)$ is a base atom, that is, an atom not containing a power of x. Rule I removes the first k of these atoms from the trial solution. Rule II removes the last $n - k$ of these atoms. The ones removed are called **superfluous atoms**.

An Illustration

Assume the differential equation has order $n = 2$ and the trial solution contains a sub-list of related atoms

$$e^{2x},\ xe^{2x},\ x^2e^{2x},\ x^3e^{ex}.$$

Example 1

Assume e^{2x} is **not** a solution of the homogeneous equation.

Then Rule I removes no atoms ($k = 0$) and Rule II removes the last 2 atoms ($n - k = 2 - 0 = 2$), resulting in the revised **shorter** atom sub-list

$$e^{2x},\ xe^{2x}.$$

Example 2

Assume e^{2x} **is** a solution of the homogeneous equation.

Then Rule I removes atom e^{2x} ($k = 1$) from the start of the list and Rule II removes x^3e^{2x} from the end of list ($n - k = 2 - 1 = 1$), resulting in the revised sub-list

$$xe^{2x},\ x^2e^{2x}.$$

Observations

- Rule I and Rule II together imply that exactly $\boldsymbol{n}$ atoms are removed from every complete sub-list of related atoms in the original trial solution.
- The $\boldsymbol{n}$ atoms are removed from *the two ends*, killing $\boldsymbol{k}$ from the *beginning* of the list and $\boldsymbol{n-k}$ from the *end* of the list.
- Substitution of the trial solution into the differential equation creates a the system of linear algebraic equations for the undetermined coefficients $\boldsymbol{d_1}$, $\boldsymbol{d_2}$, $\boldsymbol{d_3}$, …, in which **every symbol $\boldsymbol{d_j}$ appears**! There are **no free variables** and the total number of atoms used in $\boldsymbol{y}$ cannot be reduced.
- The system of equations has the least possible dimension and a unique solution for the undetermined coefficients.

A Shortcut

Building the atom list from $g(x) = x^n f(x)$ requires subsequent **removal of n atoms** from each sub-list of related atoms. Building a short atom list from $f(x)$ requires a subsequent **append of atoms** to each sub-list of related atoms. The second method, which requires less writing, is a **shortcut** recommended after learning the basic method of removing atoms.

The idea for appending the atoms is the realization that the factor x^n used in $g(x) = x^n f(x)$ causes n extra atoms to appear in a sub-list of related atoms. Here are the facts:

- If the first atom in the sublist, base atom B, is a solution of the homogeneous differential equation, then it is removed. This causes the first of the n appended atoms to be kept.
- If the first two atoms B, xB are solutions of the homogeneous differential equation, then both are removed. This causes the first two of the n appended atoms to be kept.
- If the first three atoms B, xB, x^2B are solutions of the homogeneous differential equation, then all three are removed. This causes the first three of the n appended atoms to be kept.

A Shortcut for Correction Rule II

Let a sub-list of related atoms be constructed from $f(x)$ instead of $g(x) = x^n f(x)$.

Each removal of an atom from the left causes an append of a related atom on the right.

An Example for $f(x) = 11.578x^3e^x + 22.1\cos 2x$

Consider the sub-list constructed from atom x^3e^x. The other atom $\cos 2x$ is treated similarly. Assume $n = 3$ and e^x, xe^x are homogeneous DE solutions.

Long sub-list from $x^n f(x)$	e^x	xe^x	x^2e^x	x^3e^x	x^4e^x	x^5e^x	x^6e^x
Short sub-list from $f(x)$	e^x	xe^x	x^2e^x	x^3e^x			
Remove one on the left	$\boxed{e^x}$	xe^x	x^2e^x	x^3e^x			
Append one on the right	$\boxed{}$	xe^x	x^2e^x	x^3e^x	$\boxed{x^4e^x}$		
Remove one more from the left	$\boxed{}$	$\boxed{xe^x}$	x^2e^x	x^3e^x	x^4e^x		
Append one more on the right	$\boxed{}$	$\boxed{}$	x^2e^x	x^3e^x	x^4e^x	$\boxed{x^5e^x}$	
Corrected list			x^2e^x	x^3e^x	x^4e^x	x^5e^x	