Numerical Methods : Euler, Heun, RK4

Solve $\frac{dy}{dx} = y - x - 1$, $y(0)=1$, explicitly

- $y' - y = -x - 1$ — std linear form $y' + py = q$
- $\frac{(e^{-x}y)'}{e^{-x}} = -x - 1$ — Replace left side by $(e^{P}y)'/e^{P}$ where $P = \int p$
- $\int (e^{-x}y)' = -\int (x+1)e^{-x}\,dx$ — Clear fractions; quadrature
- $e^{-x}y = (2+x)e^{-x} + c$ — Tables

  $y = ce^{x} + 2 + x$ — simplify, isolate y
- $1 = c + 2$ — substitute $x=0, y=1$ to find c [used $y(0)=1$].

$$\boxed{y = 2 + x - e^{x}}$$

Checks with text answer.

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p119 #5 Apply Euler's method to $y' = y - x - 1$, $y(0)=1$ with step sizes $h = 0.1$ and $h = 0.05$ to evaluate $y(0.5)$.

Show one step; rest by calculator or computer.

$y_1 = y_0 + h\,f(x_0, y_0)$ — Euler's formula

$y_1 = y_0 + h(y_0 - x_0 - 1)$ — use $f(x,y) = y - x - 1$

$x_0 = 0$

$y_0 = 1$ — From $y(0) = 1$

$x_1 = h$ — First step

$y_1 = 1 + h(1 - 0 - 1)$

$= 1$ — All done

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cont.

$x_0 = h$
$y_0 = 1$

Replace x_0, y_0 by first approx values found above.

$x_1 = x_0 + h$
$= 2h$

$y_1 = y_0 + h(y_0 - x_0 - 1)$
$= 1 + h(1 - h - 1)$
$= 1 - h^2$

second step done

These calculations check the first two levels of computation for Euler's method when $h = 0.1$ and $h = 0.05$. Final answers:

$y(0.5) = 0.8513$ Exact
$y(0.5) \cong 0.8895$ Euler $h = 0.1$
$y(0.5) \cong 0.8713$ Euler $h = 0.05$

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Apply Heun's method to $y' = y - x - 1$, $y(0) = 1$ with step size $h = 0.1$ to approximate $y(0.5)$.

$u = y_0 + h f(x_0, y_0)$ Heun predictor
$= y_0 + h(y_0 - x_0 - 1)$ use $f(x,y) = y - x - 1$

$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, u)]$ Heun corrector
$= y_0 + \frac{h}{2}[y_0 - x_0 - 1 + u - x_0 - h - 1]$
$= y_0 + h[y_0 - x_0 - 1] + \frac{h}{2}[h(y_0 - x_0 - 1) - h]$
$= y_0 + (h + \frac{h^2}{2})[y_0 - x_0 - 1] - \frac{h^2}{2}$

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cont

when $h = 0.1$ then

$x_0 = 0$ From $y(0) = 1$
$y_0 = 1$

$$x_1 = x_0 + h = 0.1$$

$$y_1 = y_0 + 0.105[y_0 - x_0 - 1] - 0.005$$
$$= 1 + 0.105[1 - 0 - 1] - 0.005$$
$$= 0.995$$

all done, first step

Final values are

$y(0.5) = 0.8513$ actual
$y(0.5) \cong 0.8526$ Heun approx

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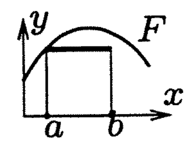
#5

Apply the RK4 method (Runge-Kutta 4th order) to $y' = y - x - 1$, $y(0) = 1$ with $h = 0.25$ to approximate $y(0.5)$.

It is too complicated to produce separate formulas for k_1, k_2, k_3, k_4. Hand calculation is not so easy, either. It is suggested that you use a computer on this problem, but summarize the steps as follows:

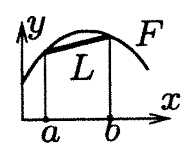
$h = 0.25$, $x_0 = 0$, $y_0 = 1$ start values
Show results for k_1, k_2, k_3, k_4
Report values of x_1, y_1.

Rectangular Rule. The approximation uses Euler's idea of replacing the integrand by a constant. The value of the integral is approximately the area of a rectangle of width $b - a$ and height $F(a)$.



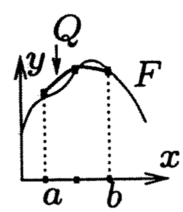
$$\int_a^b F(x)dx \approx (b-a)F(a). \tag{1}$$

Trapezoidal Rule. The rule replaces the integrand $F(x)$ by a linear function $L(x)$ which connects the planar points $(a, F(a))$, $(b, F(b))$. The value of the integral is approximately the area under the curve L, which is the area of a trapezoid.



$$\int_a^b F(x)dx \approx \frac{b-a}{2}\left(F(a) + F(b)\right). \tag{2}$$

Simpson's Rule. The rule replaces the integrand $F(x)$ by a quadratic polynomial $Q(x)$ which connects the planar points $(a, F(a))$, $((a+b)/2, F((a+b)/2))$, $(b, F(b))$. The value of the integral is approximately the area under the quadratic curve Q.



$$(3) \quad \int_a^b F(x)dx \approx \frac{b-a}{6}\left(F(a) + 4F((a+b)/2) + F(b)\right).$$

Simpson's Polynomial Rule. If $Q(x)$ is a linear, quadratic or cubic polynomial, then

$$(4) \quad \int_a^b Q(x)dx = \frac{b-a}{6}\left(Q(a) + 4Q((a+b)/2) + Q(b)\right).$$

Integrals of linear, quadratic and cubic polynomials can be evaluated *exactly* using Simpson's polynomial rule (4).