2.6 Kinetics

Studied are the following topics.

- Newton's Laws
- Free Fall with Constant Gravity
- Air Resistance Effects
- Modeling
- Parachutes
- Lunar Lander
- Escape Velocity

Newton's Laws

The ideal models of a particle or *point mass* constrained to move along the x-axis, or the motion of a projectile or satellite, have been studied from **Newton's second law**

$$F = ma. \tag{1}$$

In the *mks system* of units, F is the force in **Newtons**, m is the mass in kilograms and a is the acceleration in meters per second per second.

The closely-related **Newton universal gravitation law**

$$F = G\frac{m_1 m_2}{R^2} \tag{2}$$

is used in in conjunction with (1) to determine the system's constant value g of gravitational acceleration. The masses m_1 and m_2 have centroids at a distance R. For the earth, $g = 9.8$ m/s^2 is commonly used; see Table 1.

Other commonly used unit systems are *cgs* and *fps*. Table 1 shows some useful equivalents.

Table 1. Units for *fps* and *mks* systems

Unit name	*fps* unit	*mks* unit
Position	foot (ft)	meter (m)
Time	seconds (s)	seconds (s)
Velocity	feet/sec	meters/sec
Acceleration	feet/sec^2	meters/sec^2
Force	pound (lb)	Newton (N)
Mass	slug	kilogram (kg)
g	32.088 ft/s^2	9.7805 m/s^2

Other units in the various systems are in daily use. Table 2 shows some equivalents. An international synonym for **pound** is **libre**, with abbreviation **lb**. The origin of the word *pound* is migration of **libra pondo**, meaning *a pound in weight*. Dictionaries cite migrations *libra pondo* $\longrightarrow$ *pund* for German language, which is similar to English *pound*.

Table 2. Conversions for the *fps* and *mks* systems

inch (in)	1/12 foot	2.54 centimeters
foot (ft)	12 inches	30.48 centimeters
centimeter (cm)	1/100 meter	0.39370079 inches
kilometer (km)	1000 meters	0.62137119 miles ($\approx 5/8$)
mile (mi)	5280 feet	1.609344 kilometers ($\approx 8/5$)
pound (lb)	$\approx$ 4.448 Newtons	
Newton (N)	$\approx$ 0.225 pounds	
kilogram (kg)	$\approx$ 0.06852 slugs	
slug	$\approx$ 14.59 kilograms	

Velocity and Acceleration

The position, velocity and acceleration of a particle moving along an axis are functions of time t. Notations vary; this text uses the following symbols, where primes denote t-differentiation.

$x = x(t)$	Particle **position** at time t.
$v = x'(t)$	Particle **velocity** at time t.
$a = x''(t)$	Particle **acceleration** at time t.
$x(0)$	**Initial position**.
$v(0)$	**Initial velocity**. Synonym $x'(0)$ is also used.

Free Fall with Constant Gravity

A body falling in a constant gravitational field might ideally move in a straight line, aligned with the gravitational vector. A typical case is the *lunar lander*, which falls freely toward the surface of the moon, its progress downward controlled by retrorockets. *Falling bodies*, e.g., an object launched up or down from a tall building, can be modeled similarly. For such ideal cases, in which air resistance and other external forces are ignored, the acceleration of the body is assumed to be a constant g and the differential equation model is

$$x''(t) = -g, \quad x(0) = x_0, \quad x'(0) = v_0. \tag{3}$$

The initial position x_0 and the initial velocity v_0 must be specified. The value of g in *mks* units is $g = 9.8$ m/s^2. The symbol x is the distance from the ground ($x = 0$); meters for *mks* units. The symbol t is the time in seconds. Falling body problems normally take $v_0 = 0$ and $x_0 > 0$, e.g., x_0 is the height of the building from which the body was dropped. Objects ejected downwards have $v_0 < 0$, which decreases the descent time. Objects thrown straight up satisfy $v_0 > 0$.

Equation (3) can be solved by the method of quadrature to give the explicit solution

$$x(t) = -\frac{g}{2}t^2 + x_0 + v_0 t. \tag{4}$$

See *Technical Details*, page 137, and the *method of quadrature*, page 74. Applications to free fall and the lunar lander appear in the examples, page 132.

Typical plots can be made by the following `maple` code.

```
X:=unapply(-9.8*t^2+100+(50)*t,t); #v(0)=50m/s,x(0)=100m
plot(X(t),t=0..7);
Y:=unapply(-9.8*t^2+100+(-5)*t,t); #v(0)=-5m/s,x(0)=100m
plot(Y(t),t=0..4);
```

Air Resistance Effects

The inclusion in a differential equation model of terms accounting for air resistance has historically two distinct models. The first is *linear resistance*, in which the force F due to air resistance is assumed to be proportional to the velocity v:

$$F \propto v. \tag{5}$$

It is known that linear resistance is appropriate only for slowly moving objects. The second model is *nonlinear resistance*, modeled originally by Sir Isaac Newton himself as $F = kv^2$. The literature considers a generalized nonlinear resistance assumption

$$F \propto v|v|^p \tag{6}$$

where $0 < p \leq 1$ depends upon the *speed* of the object through the air; $p \approx 0$ is a low speed and $p \approx 1$ is a high speed. It will suffice for illustration purposes to treat just the two cases $F \propto v$ and $F \propto v|v|$.

Linear Air Resistance. The model is determined by the sum of the forces due to air resistance and gravity, $F_{\text{air}} + F_{\text{gravity}}$, which by *Newton's second law* must equal $F = mx''(t)$, giving the differential equation

$$mx''(t) = -kx'(t) - mg. \tag{7}$$

In (7), the velocity is $v = x'(t)$ and k is a proportionality constant for the air resistance force $F \propto v$. The negative sign results from the assumed coordinates: x measures the distance from the ground ($x = 0$). We expect x to decrease, hence x' is negative. Equation (7) written in terms of the velocity $v = x'(t)$ becomes

$$v'(t) = -(k/m)v(t) - g. \tag{8}$$

This equation has a solution $v(t)$ which limits at $t = \infty$ to a **finite terminal velocity** $|v_\infty| = mg/k$; see (9) below and *Technical Details*, page 137. Physically, this limit is the **equilibrium solution** of (8), which is the observable steady state of the model. A quadrature applied to $x'(t) = v(t)$ solves (7). Then

$$\begin{aligned} v(t) &= -\frac{mg}{k} + \left(v(0) + \frac{mg}{k}\right) e^{-kt/m}, \\ x(t) &= x(0) - \frac{mg}{k}t + \frac{m}{k}\left(v(0) + \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right). \end{aligned} \tag{9}$$

Nonlinear Air Resistance. The model, which applies primarily to rapidly moving objects, is obtained by the same method as the linear model, replacing the linear resistance term $kx'(t)$ by the nonlinear term $kx'(t)|x'(t)|$. The resulting model is

$$mx''(t) = -kx'(t)|x'(t)| - mg, \tag{10}$$

which in terms of the velocity $v = x'(t)$ is the first order equation

$$v'(t) = -(k/m)v(t)|v(t)| - g. \tag{11}$$

The model applies in particular to parachute flight and to certain projectile problems, like an arrow or bullet fired straight up.

Upward Launch. Separable equation (11) in the case $v(0) > 0$ for a launch upward becomes $v'(t) = -(k/m)v^2(t) - g$. The solution for $v(0) > 0$ is given below in (12); see *Technical Details*, page 137. The equation $x'(t) = v(t)$ can be solved by quadrature. Then for some constants c and d

$$\begin{aligned} v(t) &= \sqrt{\frac{mg}{k}} \tan\left(\sqrt{\frac{kg}{m}}(c - t)\right), \\ x(t) &= d + \frac{m}{k} \ln\left|\cos\left(\sqrt{\frac{kg}{m}}(c - t)\right)\right|. \end{aligned} \tag{12}$$

Downward Launch. The case $v(0) < 0$ for an object launched downward or dropped will use the equation $v'(t) = (k/m)v^2(t) - g$; see *Technical Details*, page 138. Then for some constants c and d

$$
\begin{aligned}
v(t) &= \sqrt{\frac{mg}{k}}\,\tanh\left(\sqrt{\frac{kg}{m}}(c-t)\right), \\
x(t) &= d - \frac{m}{k}\,\ln\left|\cosh\left(\sqrt{\frac{kg}{m}}(c-t)\right)\right|.
\end{aligned}
\tag{13}
$$

The **hyperbolic functions** appearing in (13) are *defined by*

$\cosh u = \frac{1}{2}\left(e^u + e^{-u}\right)$	Hyperbolic cosine.
$\sinh u = \frac{1}{2}\left(e^u - e^{-u}\right)$	Hyperbolic sine.
$\tanh u = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$	Hyperbolic tangent. Identity $\tanh u = \sinh u/\cosh u$.

The model applies to parachute problems in particular. Equation (13) and the limit formula $\lim_{|x|\to\infty}\tanh x = 1$ imply a *terminal velocity*

$$|v_\infty| = \sqrt{\frac{mg}{k}}.$$

The value is exactly the square root of the linear model terminal velocity. Without air resistance effects, e.g., the falling body model (3), the velocity is allowed to increase to unrealistic speeds.

Modeling Remarks

It can be argued from air resistance models that projectiles spend more time falling to the ground than they spend reaching maximum height[4]; see Example 32. Simplistic models ignoring air resistance tend to overestimate the maximum height of the projectile and the flight time; see Example 31. Falling bodies are predicted by air resistance models to have a *terminal velocity*.

Significant effects are ignored by the models of this text. Real projectiles are affected by spin and a flight path that is not planar. The **corkscrew** path of a bullet can cause it to miss a target, while a planar model predicts it will hit the target. The spin of a projectile can drastically alter its flight path and flight characteristics, as is known by players of

[4]Racquetball, badminton, Lacrosse, tennis, squash, pickleball and table tennis players know about this effect and use it in their game tactics and timing.

table tennis, squash and court tennis, archery enthusiasts and gun club members.

Gravitational effects assumed constant may in fact not be constant along the flight path. This can happen in the soft touchdown problem for a lunar lander, when the lander activates retrorockets high above the moon's surface.

External effects like wind or the gravitational forces of nearby celestial bodies, ignored in simplistic models, may indeed produce significant effects. On the freeway, is it possible to throw an ice cube out the window ahead of your vehicle? Is it feasible to use forces from the moon to **assist** in the launch of an orbital satellite?

Parachutes

In a typical parachute problem, the jumper travels in a parabolic arc to the ground, buffeted about by up and down drafts in the atmosphere, but always moving in the direction determined by the airplane's flight. In short, a parachutist does not *fall* to the ground. Their flight path more closely resembles the path of a projectile, but it is generally not planar.

Important to skydivers is an absolute limit to their speed, called the **terminal velocity**. It depends upon a number of physical factors, the dominant factor being body shape. A parachutist with excess loose clothing will dive more slowly than when equipped with a tight lycra jump suit. When the parachute opens, the flight characteristics are dominated by physical factors of the open parachute.

The constant $k/m > 0$ is called the **drag coefficient**, where m is the mass and $k > 0$ appears in the resistive force equation $F = kv|v|$. In order for the parachute model to give a terminal velocity of 15 miles per hour, the drag coefficient must be approximately $k/m = 3/2$. Without the parachute, the skydiver can reach speeds of over 45 miles per hour, which corresponds to a drag coefficient $k/m < 1/2$.

Who falls the greatest distance after 30 seconds, a 250-pound or a 110-pound parachutist? The answer is not so easy, because the 110-pound parachutist has *less* air resistance due to less body surface area but also *less* mass, making it difficult to compare the two drag coefficients. A layman's answer might be serendipitously correct!

Lunar Lander

A lunar lander is falling toward the moon's surface, in the radial direction, at a speed of 1000 miles per hour. It is equipped with retrorockets to retard the fall. In free space outside the gravitational effects of the moon the retrorockets provide a retardation thrust of 9 miles per hour per second of activation, e.g., 11 seconds of retrorocket power will slow the lander down by about 100 miles per hour.

A **soft touchdown** is made when the lander contacts the moon's surface falling at a speed of zero miles per hour. This ideal situation can be achieved by turning on the retrorockets at the right moment.

The lander is greatly affected by the gravitational field of the moon. Ignoring this field gives a gross overestimate for the activation time, causing the lander to reverse its direction and never reach the surface. The layman answer of $1000/9 \approx 112$ seconds to touchdown from an altitude of about 16 miles is incorrect by about 10 miles, causing the lander to crash at substantial speed into the lunar surface.

Escape velocity

Is it possible to fire a projectile from the earth's surface and reach the moon? The science fiction author Jules Verne, in his 1865 novel *From the Earth to the Moon*, seems to believe it is possible. Modern calculations give the initial **escape velocity** v_0 as about $25,000$ miles per hour. There is no record of this actually being tested, so the number $25,000$ remains a theoretical estimate.

This is a different problem than powered rocket flight. All the power must be applied initially, and it is not allowed to apply power during flight to the moon. Imagine instead a deep hole, in which a rocket is launched, the power being turned off just as the rocket exits the hole. The rocket has to coast to the moon, using just the velocity gained during launch.

Newton's law of universal gravitation gives m_1m_2G/r^2 as the magnitude of the force of attraction between two point-masses m_1, m_2 separated by distance r. The equation $g = Gm_2/R^2$ gives the acceleration due to gravity at the surface of the planet. For the earth, $g = 9.8$ meters per second per second and $R = 6,370,000$ meters.

A spherical projectile of mass m_1 hurled straight up from the surface of a planet moves in the radial direction. Ignoring air resistance and external gravitational forces, Newton's law implies the distance $y(t)$ traveled by the projectile satisfies

$$(14) \qquad m_1y''(t) = -\frac{m_1m_2G}{(y(t)+R)^2}, \quad y(0) = 0, \quad y'(0) = v_0,$$

where R is the radius of the planet, m_2 is its mass and G is the experimentally measured universal gravitation constant. Using $gR^2 = Gm_2$ and canceling m_1 in (14) gives

$$(15) \qquad y''(t) = -\frac{gR^2}{(y(t)+R)^2}, \quad y(0) = 0, \quad y'(0) = v_0.$$

The projectile **escapes** the planet if $y(t) \to \infty$ as $t \to \infty$. The **escape velocity problem** asks which minimal value of v_0 causes escape.

To solve the escape velocity problem, multiply equation (15) by $y'(t)$, then integrate over $[0,t]$ and use the initial conditions $y(0) = 0$, $y'(0) = v_0$ to obtain

$$\frac{1}{2}\left((y'(t))^2 - (v_0)^2\right) = \frac{gR^2}{y(t)+R} - Rg.$$

The square term $(y'(t))^2$ being nonnegative gives the inequality

$$0 \le (v_0)^2 + \frac{2gR^2}{y(t)+R} - 2Rg.$$

If $y(t) \to \infty$, then $v_0^2 \ge 2Rg$, which gives the *escape velocity*

$$v_0 = \sqrt{2gR}. \tag{16}$$

For the earth, $v_0 \approx 11,174$ meters per second, which is slightly more than $25,000$ miles per hour.

Examples

29 **Example (Free Fall)** A ball is thrown straight up from the roof of a 100-foot building and allowed to fall to the ground. Assume initial velocity $v_0 = 32$ miles per hour. Estimate the maximum height of the ball and its flight time to the ground.

Solution: The maximum height H and flight time T are given by

$$H = 134.41 \text{ ft}, \quad T = 4.36 \text{ sec}.$$

Details: In fps units, $v_0 = 32(5280)/(3600) = 46.93$ ft/sec. Using solution (4) gives for $x_0 = 100$ and $v_0 = 46.93$

$$x(t) = -16t^2 + 100 + 46.93t.$$

Then $x(t) = H = \max$ when $x'(t) = 0$, which happens at $t = 46.93/32$. Therefore, $H = x(46.93/32) = 134.41$. The flight time T is given by the equation $x(T) = 0$ (the ground is $x = 0$). Solving this quadratic equation for $T > 0$ gives $T = 4.36$ seconds.

30 **Example (Lunar Lander)** A lunar lander falls to the moon's surface at $v_0 = -960$ miles per hour. The retrorockets in free space provide a deceleration effect on the lander of $a = 18,000$ miles per hour per hour. Estimate the retrorocket activation height above the surface which will give the lander zero touch-down velocity.

Solution: Presented here are two models, one which assumes the moon's gravitational field is constant and another which assumes it is variable. The results obtained for the activation height are different: 93.3 miles for the constant field model and 80.1 miles for the variable field model. The flight time to touchdown for the constant field model is estimated to be 11.7 minutes and for the variable field model the estimate is 10.4 minutes.

The *mks* unit system will be used for calculations, giving $v_0 = -429.1584$ meters per second and $a = 2.2352$ meters per second per second.

Constant field model. Let's assume constant gravitational acceleration $\mathcal{G}$ due to the moon. Other gravitational effects are ignored.

The acceleration value $\mathcal{G}$ is found in *mks* units from the formula

$$\mathcal{G} = \frac{Gm_1}{R^2}.$$

Here, $m_1 = 7.36 \times 10^{22}$ kilograms and $R = 1.74 \times 10^6$ meters (1740 kilometers, 1081 miles) are the mass and radius of the moon. Newton's universal gravitation constant is $G \approx 6.6726 \times 10^{-11}$ N(m/kg)^2. Then $\mathcal{G} = 1.621942132$.

The lander itself has mass m. Let $r(t)$ be the distance from the lander to the surface of the moon. The value $r(0)$ is the height above the moon when the retrorockets are activated for the soft landing at time t_0. Then force analysis and Newton's second law implies the model

$$mr''(t) = ma - m\mathcal{G}, \quad r(t_0) = 0, \quad r'(t_0) = 0, \quad r'(0) = v_0.$$

The objective is to find $r(0)$. Cancel m, then integrate twice to obtain the quadrature solution

$$\begin{array}{rcl} r'(t) & = & (a - \mathcal{G})t + v_0, \\ r(t) & = & (a - \mathcal{G})t^2/2 + v_0 t + r(0). \end{array}$$

Then $r'(t_0) = 0$ and $r(t_0) = 0$ give the equations

$$(a - \mathcal{G})t + v_0 = 0, \quad r(0) = -v_0 t_0 - (a - \mathcal{G})t_0^2/2.$$

Evaluation uses *mks* units: $a = 2.2352$, $v_0 = -429.1584$, $\mathcal{G} = 1.621942132$. Solving simultaneously provides the numerical answers

$$t_0 = 11.66 \text{ minutes}, \quad r(0) = 150.16 \text{ kilometers} = 93.3 \text{ miles}.$$

The conversion uses 1 mile = 1.609344 kilometers.

Variable field model. The constant field model will be modified to obtain this model. All notation developed above applies. We will replace the constant acceleration $\mathcal{G}$ by the variable acceleration $Gm_1/(R + r(t))^2$. Then the model is

$$mr''(t) = ma - \frac{Gmm_1}{(R + r(t))^2}, \quad r(t_0) = 0, \quad r'(t_0) = 0, \quad r'(0) = v_0.$$

Multiply this equation by $r'(t)/m$ and integrate. Then

$$\frac{(r'(t))^2}{2} = ar(t) + \frac{Gm_1}{R + r(t)} + c, \quad c \equiv -\frac{Gm_1}{R}.$$

We want to find $r(0)$, the height above the moon. The equation to solve for $r(0)$ is found by substitution of $t = 0$ into the previous equation:

$$\frac{(r'(0))^2}{2} = ar(0) + \frac{Gm_1}{R + r(0)} - \frac{Gm_1}{R}.$$

After substitution of known values, the quadratic equation for $x = r(0)$ is given by

$$92088.46615 = 2.2352x + \frac{2822179.310}{1 + x/1740000} - 2822179.310.$$

Solving for the positive root gives $r(0) \approx 127.23$ kilometers or 79.06 miles. The analysis does not give the flight time t_0 directly, but it is approximately 10.4 minutes: see the exercises.

Answer check. A similar analysis is done in Edwards and Penney [**?**] for the case $a = 4$ meters per second per second, $v_0 = -450$ meters per second, with result $r(0) \approx 41.87$ kilometers. In their example, the retrorocket thrust is nearly doubled, resulting in a lower activation height. The reader can substitute $v_0 = -450$ and $a = 4$ in the variable field model to obtain agreement: $r(0) \approx 41.90$ kilometers. The constant field model gives $r(0) \approx 42.58$ kilometers and $t_0 \approx 3.15$ minutes.

31 Example (Flight Time and Maximum Height) Show that the maximum height and the ascent time of a projectile are over-estimated by a model that ignores air resistance.

Solution: Treated here is the case of a projectile launched straight up from the ground $x = 0$ with velocity $v_0 > 0$. The ascent time is denoted t_1 and the maximum height M is then $M = x(t_1)$.

No air resistance. Consider the model $v' = -g$, $v(0) = v_0$. The solution is $v = -gt + v_0$, $x = -gt^2/2 + v_0 t$. Then maximum height M occurs at $v'(t_1) = 0$ which gives $t_1 = v_0/g$ and $M = x(t_1) = t_1(v_0 - gt_1/2) = gv_0^2/2$.

Linear air resistance. Consider the model $v' = -\rho v - g$, $v(0) = v_0$. This is a Newton cooling equation in disguise, with solution given by equation (9), where $\rho = k/m$. Then t_1 is a function of (ρ, v_0) satisfying $ge^{\rho t_1} = v_0\rho + g$, hence t_1 is given by the equation

$$t_1(\rho, v_0) = \frac{1}{\rho}\ln\left|\frac{v_0\rho + g}{g}\right|. \tag{17}$$

The limit of $t_1 = t_1(\rho, v_0)$ as $\rho \to 0$ is the ascent time v_0/g of the no air resistance model. We verify in the exercises the following.

Lemma 2 (Linear Ascent Time) The ascent time t_1 for linear air resistance satisfies $t_1(\rho, v_0) < v_0/g$.

The lemma implies that the rise time for linear air resistance is less than the rise time for no air resistance.

The inequality $v' = -\rho v - g < -g$ holds for $v > 0$, therefore $v(t) < -gt + v_0$ and $x(t) < -gt^2/2 + v_0 t =$ height for the no air resistance model. Thus the maximum height $x(t_1)$ is less than the maximum height for the no air resistance model, by Lemma 2; see the exercises page 142.

Nonlinear air resistance. We are technically done with the example, since it has been shown that the answers for t_1 and M decrease when using the linear model. Similar results can be stated for the nonlinear model $v' = \rho v|v| - g$; see the exercises page 142.

32 **Example (Modeling)** Argue from nonlinear air resistance models that a projectile takes more time to fall to the ground than it takes to reach maximum height.

Solution: The model will be the nonlinear model of the text, which historically goes back to Newton himself. The linear air resistance model, appropriate for slowly moving projectiles, is not considered in this example.

Let t_1 and t_2 be the ascent and fall times, so that the total flight time from the ground to maximum height and then to the ground again is $t_1 + t_2$.

The times t_1, t_2 are functions of the initial velocity $v_0 > 0$. As v_0 limits to zero, both t_1 and t_2 limit to zero. We derive $t_2 dt_2/dv_0 - t_1 dt_1/dv_0 > 0$ in Lemma 7 below. Integrating the inequality on variable v_0 then implies $\frac{1}{2}(t_2^2 - t_1^2) > 0$, from which it follows that $t_2 > t_1$ for $v_0 > 0$. This means that the projectile takes more time to fall to the ground (t_2) than it takes to reach maximum height (t_1).

Let $f_1(v) = -(k/m)v^2 - g$ and $f_2(v) = (k/m)v^2 - g$.

The **ascent** is controlled with velocity $v_1 > 0$ satisfying $v_1' = f_1(v_1)$, $v_1(0) = v_0 > 0$, $v_1(t_1) = 0$. The maximum height reached is $y_0 = \int_0^{t_1} v_1(t)dt$. The **descent** is controlled with velocity $v_2(t)$ satisfying $v_2' = f_2(v_2)$, $v_2(t_1) = 0$. The flight ends at time $T = t_1 + t_2$, determined by $0 = y_0 + \int_{t_1}^{T} v_2(t)dt$.

The details of proof involve a number of technical results, some of which depend upon the formulas for the nonlinear functions f_1, f_2.

Lemma 3 The solution v_2 satisfies $v_2(t) = w(t - t_1)$, where w is defined by $w' = f_2(w)$, $w(0) = 0$. The solution w does not involve variables v_0, t_1, t_2.

Lemma 4 Assume f is continuously differentiable. Let $v(t, v_0)$ be the solution of $v' = f(v)$, $v(0) = v_0$. Then

$$\frac{dv}{dv_0} = e^{\int_0^t f'(v(t,v_0))dt}.$$

The function $z = dv/dv_0$ solves the linear problem $z' = f'(v(t, v_0))z$, $z(0) = 1$.

Lemma 5

$$\frac{dt_1}{dv_0} = \frac{1}{g} e^{-2k \int_0^{t_1} v_1(t,v_0)dt/m}.$$

Lemma 6

$$\frac{dt_2}{dv_0} = \frac{-1}{v_2(t_1 + t_2)} \int_0^{t_1} e^{-2k \int_0^t v_1(r,v_0)dr/m}\, dt.$$

Lemma 7

$$t_2 \frac{dt_2}{dv_0} - t_1 \frac{dt_1}{dv_0} > 0.$$

Proof of Lemma 7. Lemmas 3 to 6 will be applied. Define $w(t)$ by Lemma 3. Because $w' = f_2(w) = (k/m)w^2 - g$, then $f_2(w) \ge -g$ which implies $w(t) \ge w(0) - gt$. Using $w(0) = 0$ implies $v_2(t_1 + t_2) = w(t_2) \ge -gt_2$ and finally, using $w(t) < 0$ for $0 < t \le t_2$,

$$\frac{1}{gt_2} \le \frac{-1}{v_2(t_1 + t_2)}.$$

Multiply this inequality by $e^{u(t)}$, $u(t) = -2k \int_0^t v_1(r, v_0)dr/m$. Integrate over $t = 0$ to $t = t_1$. Then Lemma 6 implies

$$\frac{1}{gt_2} \int_0^{t_1} e^{u(t)} dt \le \frac{dt_2}{dv_0}.$$

Because $u(t) > u(t_1)$, then

$$\frac{1}{gt_2} \int_0^{t_1} e^{u(t_1)} dt < \frac{dt_2}{dv_0}.$$

This implies by Lemma 5 the inequality

$$\frac{t_1}{t_2} \frac{dt_1}{dv_0} = \frac{t_1}{gt_2} e^{u(t_1)} < \frac{dt_2}{dv_0},$$

or $t_2 dt_2/dv_0 - t_1 dt_1/dv_0 > 0$. The proof is complete.

Proof of Lemma 3. The function $z(t) = v_2(t+t_1)$ satisfies $z' = f_2(z)$, $z(0) = 0$ (an answer check for the reader). Function $w(t)$ is defined to solve $w' = f_2(w)$, $w(0) = 0$. By uniqueness, $z(t) \equiv w(t)$, or equivalently, $w(t) = v_2(t + t_1)$. Replace t by $t - t_1$ to obtain $v_2(t) = w(t - t_1)$.

Proof of Lemma 4. The exponential formula for dv_2/dv_0 is the unique solution of the first order initial value problem. It remains to show that the initial value problem is satisfied. Instead of doing the answer check, we motivate how to find the initial value problem. First, differentiate across the equation $v_2' = f_2(v_2)$ with respect to variable v_0 to obtain $z' = f_2'(v_2)z$ where $z = dv_2/dv_0$. Secondly, differentiate the relation $v_2(0, v_0) = v_0$ on variable v_0 to obtain $z(0) = 1$. The details of the answer check focus on showing Newton quotients converge to the given answer.

Proof of Lemma 5. Start with the determining equation $v_1(t_1, v_0) = 0$. Differentiate using the chain rule on variable v_0 to obtain the relation

$$v_1'(t_1, v_0)\frac{dt_1}{dv_0} + \frac{dv_1}{dv_0}(t_1, v_0) = 0.$$

Because $f_1'(u) = -2ku/m$, then the preceding lemma implies that dv_1/dv_0 is the same exponential function as in this Lemma. Also, $v_1(t_1, v_0) = 0$ implies $v_1'(t_1, v_0) = f_1(0) = -g$. Substitution gives the formula for dt_1/dv_0.

Proof of Lemma 6. Start with $y_0 = \int_0^{t_1} v_1(t, v_0)dt$ and $y(t) = y_0 + \int_{t_1}^t v_2(t)dt$. Then $0 = y(t_2 + t_1)$ implies that

$$\begin{aligned} 0 &= y(t_1 + t_2) \\ &= \textstyle\int_0^{t_1} v_1(t, v_0)dt + \int_0^{t_2} v_2(t + t_1)dt \\ &= \textstyle\int_0^{t_1} v_1(t, v_0)dt + \int_0^{t_2} w(t)dt. \end{aligned}$$

Because $w(t)$ is independent of t_1, t_2, v_0 and $v_1(t_1, v_0) = 0$, then differentiation on v_0 across the preceding formula gives

$$\begin{aligned} 0 &= \frac{d}{dv_0}\int_0^{t_1} v_1(t,v_0)dt + w(t_2)\frac{dt_2}{dv_0} \\ &= v_1(t_1,v_0)\frac{dt_1}{dv_0} + \int_0^{t_1}\frac{dv_1}{dv_0}(t,v_0)dt + w(t_2)\frac{dt_2}{dv_0} \\ &= 0 + \int_0^{t_1} e^{u(t)}dt + w(t_2)\frac{dt_2}{dv_0} \end{aligned}$$

where $u(t) = -2k\int_0^t v_1(r, v_0)dr/m$. Use $w(t_2) = v_2(t_2 + t_1)$ after division by $w(t_2)$ in the last display to obtain the formula.

Details and Proofs

Proof for Equation (4). The method of quadrature is applied as follows.

$x''(t) = -g$	The given differential equation.
$\int x''(t)dt = \int -gdt$	Quadrature step.
$x'(t) = -gt + c_1$	Fundamental theorem of calculus.
$\int x'(t)dt = \int(-gt + c_1)dt$	Quadrature step.
$x(t) = -g\frac{t^2}{2} + c_1t + c_2$	Fundamental theorem of calculus.

Using initial conditions $x(0) = x_0$ and $x'(0) = v_0$ it follows that $c_1 = v_0$ and $c_2 = x_0$. These steps verify the formula $x(t) = -gt^2/2 + x_0 + v_0t$.

Technical Details for Equation (9).

$v'(t) + (k/m)v(t) = -g$	Standard linear form.
$\frac{(Qv)'}{Q} = -g$	Integrating factor $Q = e^{kt/m}$.
$(Qv)' = -gQ$	Quadrature form.
$Qv = -mgQ/k + c$	Method of quadrature.
$v = -mg/k + c/Q$	Velocity equation.
$v = -\frac{mg}{k} + \left(v(0) + \frac{mg}{k}\right)e^{-kt/m}$	Evaluate c and use $Q = e^{kt/m}$.

The equation $x(t) = x(0) + \int_0^t v(r)dr$ gives the last relation in (9):

$$x(t) = x(0) - \frac{mg}{k}t + \frac{m}{k}\left(v(0) + \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right).$$

Technical Details for Equation (12), $v(0) > 0$.

$v'(t) = -(k/m)v^2(t) - g$	The upward launch equation.
$u'(t) = \sqrt{\frac{kg}{m}}(1 + u^2(t))$	Change of variables $u = \sqrt{\frac{k}{mg}}\,v$.
$\frac{u'(t)}{1+u^2(t)} = -\sqrt{\frac{kg}{m}}$	A separated form.
$\arctan(u(t)) = -\sqrt{\frac{kg}{m}}t + c_1$	Quadrature.
$u(t) = \tan\left(c_1 - \sqrt{\frac{kg}{m}}t\right)$	Take the tangent of both sides.

$v(t) = \sqrt{\frac{mg}{k}} \tan\left(\sqrt{\frac{kg}{m}}(c-t)\right)$	Define $c_1 = \sqrt{\frac{kg}{m}}c$.
$x(t) = \int v(t)dt$	Quadrature method.
$= d + \frac{m}{k} \ln\left\|\cos\left(\sqrt{\frac{kg}{m}}(c-t)\right)\right\|$	Integration constant d.

Technical Details for Equation (13), $v(0) < 0$.

$v'(t) = (k/m)v^2(t) - g$	Downward launch equation.
$u'(t) = \sqrt{\frac{kg}{m}}\left(u^2(t) - 1\right)$	Change of variables $u = \sqrt{\frac{k}{mg}}\,v$.
$\frac{u'(t)}{u^2(t)-1} = \sqrt{\frac{kg}{m}}$	A separated form.
$-\operatorname{arctanh}(u) = 2t\sqrt{\frac{kg}{m}} + c_1$	Quadrature method and tables.
$u = \tanh\left(\sqrt{\frac{kg}{m}}(c-t)\right)$	Define c by $\sqrt{\frac{kg}{m}}\,c = -c_1$.
$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}(c-t)\right)$	Use $v = \sqrt{\frac{mg}{k}}\,u$.
$x(t) = \int v(t)dt$	Quadrature.
$= d - \frac{m}{k} \ln\left\|\cosh\left(\sqrt{\frac{kg}{m}}(c-t)\right)\right\|$	Integration constant d.

Exercises 2.6

Newton's Laws. Review of units and conversions.

1. An object weighs 100 pounds. Find its mass in slugs and kilograms.

2. An object has mass 50 kilograms. Find its mass in slugs and its weight in pounds.

3. Convert from *fps* to *mks* systems: position 1000, velocity 10, acceleration 2.

4. Derive $g = \dfrac{Gm}{R^2}$, where m is the mass of the earth and R is its radius.

Velocity and Acceleration. Find the velocity x' and acceleration x''.

5. $x(t) = 16t^2 + 100$

6. $x(t) = 16t^2 + 10t + 100$

7. $x(t) = t^3 + t + 1$

8. $x(t) = t(t-1)(t-2)$

Free Fall with Constant Gravity. Solve using the model $x''(t) = -g$, $x(0) = x_0$, $x'(0) = v_0$.

9. A brick falls from a tall building, straight down. Find the distance it fell and its speed at three seconds.

10. An iron ingot falls from a tall building, straight down. Find the distance it fell and its speed at four seconds.

11. A ball is thrown straight up from the ground with initial velocity 66 feet per second. Find its maximum height.

12. A ball is thrown straight up from the ground with initial velocity 88 feet per second. Find its maximum height.

13. An arrow is shot straight up from the ground with initial velocity 23 meters per second. Find the flight time back to the ground.

14. An arrow is shot straight up from the ground with initial velocity 44 meters per second. Find the flight time back to the ground.

15. A car travels 140 kilometers per hour. Brakes are applied, with deceleration 10 meters per second per second. Find the distance the car travels before stopping.

16. A car travels 120 kilometers per hour. Brakes are applied, with deceleration 40 feet per second per second. Find the distance the car travels before stopping.

17. An arrow is shot straight down from a height of 500 feet, with initial velocity 44 feet per second. Find the flight time to the ground and its impact speed.

18. An arrow is shot straight down from a height of 200 meters, with initial velocity 13 meters per second. Find the flight time to the ground and its impact speed.

Linear Air Resistance. Solve using the linear air resistance model $mx''(t) = -kx'(t) - mg$. An equivalent model is $x'' = -\rho x - g$, where $\rho = k/m$ the drag coefficient.

19. An arrow is shot straight up from the ground with initial velocity 23 meters per second. Find the flight time back to the ground. Assume $\rho = 0.035$.

20. An arrow is shot straight up from the ground with initial velocity 27 meters per second. Find the maximum height. Assume $\rho = 0.04$.

21. A parcel is dropped from an aircraft at $32,000$ feet. It has a parachute that opens automatically after 25 seconds. Assume drag coefficient $\rho = 0.16$ without the parachute and $\rho = 1.45$ with it. Find the descent time to the ground.

22. A first aid kit is dropped from a helicopter at $12,000$ feet. It has a parachute that opens automatically after 15 seconds. Assume drag coefficient $\rho = 0.12$ without the parachute and $\rho = 1.55$ with it. Find the impact speed with the ground.

23. A motorboat has velocity v satisfying $1100v'(t) = 6000 - 110v$, $v(0) = 0$. Find the maximum speed of the boat.

24. A motorboat has velocity v satisfying $1000v'(t) = 4000 - 90v$, $v(0) = 0$. Find the maximum speed of the boat.

25. A parachutist falls until his speed is 65 miles per hour. He opens the parachute. Assume drag coefficient $\rho = 1.57$. About how many seconds must elapse before his speed is reduced to within 1% of terminal velocity?

26. A parachutist falls until his speed is 120 kilometers per hour. He opens the parachute. Assume drag coefficient $\rho = 1.51$. About how many seconds must elapse before his speed is reduced to within 2% of terminal velocity?

27. A ball is thrown straight up with initial velocity 35 miles per hour. Find the ascent time and the descent time. Assume drag coefficient 0.042

28. A ball is thrown straight up with initial velocity 60 kilometers per hour. Find the ascent time and the descent time. Assume drag coefficient 0.042

Linear Ascent and Descent Times. Find the ascent time t_1 and the descent time t_2 for the linear model $v' = -\rho v - g$, $\rho = k/m$ is the drag coefficient. Use equation (17) for t_1. Find t_2 from $x(t_2) = 0$, where $v = x'$ and $v' = -\rho v - g$, $v(0) = 0$, $x(0) = y_0$ and $y_0 = \rho^{-1} \int_0^{t_1} (-g + (v_0\rho + g)e^{-\rho t})dt$.

29. $\rho = 0.01$

30. $\rho = 0.015$

31. $\rho = 0.02$

32. $\rho = 0.018$

33. $\rho = 0.022$

34. $\rho = 0.025$

35. $\rho = 1.5$

36. $\rho = 1.55$

37. $\rho = 1.6$

38. $\rho = 1.65$

39. $\rho = 1.45$

40. $\rho = 1.48$

Nonlinear Air Resistance. Assume ascent velocity v_1 satisfies $v_1' = -\rho v_1^2 - g$. Assume descent velocity v_2 satisfies $v_2' = \rho v_2^2 - g$. Let t_1 and t_2 be the ascent and descent times, so that $t_1 + t_2$ is the flight time. Let $v_1(0) = v_0$ and $v_2(t_1) = v_1(t_1) = 0$. Units are mks. Assume $g = 9.8$. Define M = maximum height and v_f = impact velocity.

41. Let $\rho = 0.0012$, $v_0 = 50$. Find t_1, t_2

42. Let $\rho = 0.0012$, $v_0 = 30$. Find t_1, t_2

43. Let $\rho = 0.0015$, $v_0 = 50$. Find t_1, t_2

44. Let $\rho = 0.0015$, $v_0 = 30$. Find t_1, t_2

45. Let $\rho = 0.001$, $v_0 = 50$. Find M, v_f.

46. Let $\rho = 0.001$, $v_0 = 30$. Find M, v_f.

47. Let $\rho = 0.0014$, $v_0 = 50$. Find M, v_f.

48. Let $\rho = 0.0014$, $v_0 = 30$. Find M, v_f.

49. Find t_1, t_2, M and v_f for $\rho = 0.00152$, $v_0 = 60$.

50. Find t_1, t_2, M and v_f for $\rho = 0.00152$, $v_0 = 40$.

Terminal Velocity. Find the terminal velocity for (a) a linear air resistance $a(t) = \rho v(t)$ and (b) a nonlinear air resistance $a(t) = \rho v^2(t)$. Use the model equation $v' = a(t) - g$ and the given drag coefficient ρ.

51. $\rho = 0.15$

52. $\rho = 0.155$

53. $\rho = 0.015$

54. $\rho = 0.017$

55. $\rho = 1.5$

56. $\rho = 1.55$

57. $\rho = 2.0$

58. $\rho = 1.89$

59. $\rho = 0.001$

60. $\rho = 0.0015$

Parachutes. A parachute opens at timer value $t = 0$ and the body falls at speed v given by (a) linear resistance model $v' = \rho v - g$ or (b) nonlinear resistance model $v' = \rho v^2 - g$. Given the drag coefficient ρ and initial velocity $v(0) = v_0$, compute the elapsed distance and elapsed time until the body reaches 98% of its terminal velocity. Report two values for (a) and two values for (b).

61. $\rho = 1.446$, $v_0 = -66$ ft/sec.

62. $\rho = 1.446$, $v_0 = -44$ ft/sec.

63. $\rho = 1.5$, $v_0 = -66$ ft/sec.

64. $\rho = 1.5$, $v_0 = -44$ ft/sec.

65. $\rho = 1.55$, $v_0 = -21$ ft/sec.

66. $\rho = 1.55$, $v_0 = -11$ ft/sec.

67. $\rho = 1.442$, $v_0 = 0$ ft/sec.

68. $\rho = 1.442$, $v_0 = -5$ ft/sec.

69. $\rho = 1.37$, $v_0 = -44$ ft/sec.

70. $\rho = 1.37$, $v_0 = -22$ ft/sec.

Lunar Lander. A lunar lander falls to the moon's surface at v_0 miles per hour. The retrorockets in free space provide a deceleration effect on the lander of a miles per hour per hour. Estimate the retrorocket activation height above the surface which will give the lander zero touch-down velocity. Follow Example 30, page 132.

71. $v_0 = -1000$, $a = 18000$

72. $v_0 = -980$, $a = 18000$

73. $v_0 = -1000$, $a = 20000$

74. $v_0 = -1000$, $a = 19000$

75. $v_0 = -900$, $a = 18000$

76. $v_0 = -900$, $a = 20000$

77. $v_0 = -1100$, $a = 22000$

78. $v_0 = -1100$, $a = 21000$

79. $v_0 = -800$, $a = 18000$

80. $v_0 = -800$, $a = 21000$

Escape velocity. Find the escape velocity of the given planet, given the planet's mass m and radius R.

81. (Planet A) $m = 3.1 \times 10^{23}$ kilograms, $R = 2.4 \times 10^7$ meters.

82. (Mercury) $m = 3.18 \times 10^{23}$ kilograms, $R = 2.43 \times 10^6$ meters.

83. (Planet B) $m = 5.1 \times 10^{24}$ kilograms, $R = 6.1 \times 10^6$ meters.

84. (Venus) $m = 4.88 \times 10^{24}$ kilograms, $R = 6.06 \times 10^6$ meters.

85. (Mars) $m = 6.42 \times 10^{23}$ kilograms, $R = 3.37 \times 10^6$ meters.

86. (Neptune) $m = 1.03 \times 10^{26}$ kilograms, $R = 2.21 \times 10^7$ meters.

87. (Jupiter) $m = 1.90 \times 10^{27}$ kilograms, $R = 6.99 \times 10^7$ meters.

88. (Uranus) $m = 8.68 \times 10^{25}$ kilograms, $R = 2.33 \times 10^7$ meters.

89. (Saturn) $m = 5.68 \times 10^{26}$ kilograms, $R = 5.85 \times 10^7$ meters.

90. (Pluto) $m = 1.44 \times 10^{22}$ kilograms, $R = 1.5 \times 10^6$ meters.

Lunar Lander Experiments.

91. (Lunar Lander) Verify that the variable field model for Example 30 gives a soft landing flight model in MKS units

$$\begin{aligned} u''(t) &= 2.2352 - \frac{c_1}{(c_2 + u(t))^2}, \\ u(0) &= 127233.2115, \\ u'(0) &= -429.1584, \end{aligned}$$

where $c_1 = 4910591999000$ and $c_2 = 1740000$.

92. (Lunar Lander: Numerical Experiment) Using a computer, solve the flight model of the previous exercise. Determine the flight time $t_0 \approx 625.5$ seconds by solving $u(t) = 0$ for t.

Details and Proofs.

93. (Linear Rise Time) Using the inequality $e^u > 1 + u$ for $u > 0$, show that the ascent time t_1 in equation (17) satisfies

$$g(1 + \rho t_1) < g e^{\rho t_1} = v_0 \rho + g.$$

Conclude that $t_1 < v_0/g$, proving Lemma 2.

94. **(Linear Maximum)** Verify that Lemma 2 plus the inequality $x(t) < -gt^2/2 + v_0 t$ imply $x(t_1) < gv_0^2/2$. Conclude that the maximum for $\rho > 0$ is less than the maximum for $\rho = 0$.

95. **(Linear Rise Time)** Consider the ascent time $t_1(\rho, v_0)$ given by equation (17). Prove that

$$\frac{dt_1}{d\rho} = \frac{\ln \frac{g}{v0\rho+g}}{\rho^2} + \frac{v0}{\rho(v0\rho + g)}.$$

96. **(Linear Rise Time)** Consider $dt_1(\rho, v_0)/d\rho$ given in the previous exercise. Let $\rho = gx/v_0$. Show that $dt_1/d\rho < 0$ by considering properties of the function $-(x+1)\ln(x+1) + x$. Then prove Lemma 2.

97. **(Compare Rise Times)** Show that the nonlinear ascent time for the model $v' = -\rho v^2 - g$ is less than the linear ascent time from model $v' = -\rho v - g$.

98. **(Compare Fall Times)** Show that the nonlinear descent time for the model $v' = \rho v^2 - g$ is less than the linear descent time from model $v' = -\rho v - g$.