2.3 Linear Equations

An equation $y' = f(x, y)$ is called **first-order linear** or a **linear equation** provided it can be rewritten in the special form

$$y' + p(x)y = r(x) \tag{1}$$

for some functions $p(x)$ and $r(x)$. In most applications, p and r are assumed to be continuous. The function $p(x)$ is called the **coefficient of** y. The function $r(x)$ (r abbreviates *right side*) is called the **non-homogeneous term** or the **forcing term**. Engineering texts call $r(x)$ the **input** and the solution $y(x)$ the **output**.

A practical test:

> An equation $y' = f(x, y)$ with f continuously differentiable is **linear** provided $f_y(x, y)$ is independent of y.

Form (1) is obtained by defining $r(x) = f(x, 0)$ and $p(x) = -f_y(x, y)$. Two examples:

$Ly' + Ry = E$	The LR-circuit equation with $p(x) = R/L$ and $r(x) = E/L$. Symbols L, R and E are respectively inductance, resistance and electromotive force.
$y' + xy = 0$	Airy's airfoil equation with $p(x) = x$ and $r(x) = 0$.

Classifying Linear Equations

Algebraic complexity may make an equation $y' = f(x, y)$ appear to be **non-linear**, e.g., $y' = (\sin^2(xy) + \cos^2(xy))y$ simplifies to $y' = y$.

Computer algebra systems classify an equation $y' = f(x, y)$ as linear provided the identity $f(x, y) = f(x, 0) + f_y(x, 0)y$ is valid. Equivalently, $f(x, y) = r(x) - p(x)y$, where $r(x) = f(x, 0)$ and $p(x) = -f_y(x, y)$. Automatic simplifications in computer algebra systems make this test practical. Hand verification can use the same method.

Elimination of an equation $y' = f(x, y)$ from the class of linear equations can be done from *necessary conditions*. The equality $f_y(x, y) = f_y(x, 0)$ implies two such conditions:

1. If $f_y(x, y)$ depends on y, then $y' = f(x, y)$ is not linear.

2. If $f_{yy}(x, y) \neq 0$, then $y' = f(x, y)$ is not linear.

For instance, either condition implies $y' = 1 + y^2$ is *not linear*.

The Integrating Factor Method

The initial value problem

$$(2) \qquad y' + p(x)y = r(x), \quad y(x_0) = y_0,$$

where p and r are continuous in an interval containing $x = x_0$, has an explicit solution (justified on page 78)

$$(3) \qquad y(x) = e^{-\int_{x_0}^{x} p(s)ds}\left(y_0 + \int_{x_0}^{x} r(t)e^{-\int_{x_0}^{t} p(s)ds}\,dt\right).$$

Formula (3) is called **variation of parameters**, for historical reasons. While (3) has some appeal, applications use the **integrating factor method** below, which is developed with indefinite integrals for computational efficiency. No one memorizes (3); they memorize the *method*. See Example 11, page 75, for technical details.

Integrating Factor Identity. The technique called **the method of integrating factors** uses the replacement rule (justified on page 78)

$$(4) \qquad \text{The fraction} \quad \frac{\left(e^{\int p(x)dx}Y\right)'}{e^{\int p(x)dx}} \quad \text{replaces} \quad Y' + p(x)Y.$$

The factor $e^{\int p(x)dx}$ in (4) is called an **integrating factor**.

The Integrating Factor Method

Standard Form	Rewrite $y' = f(x,y)$ in the form $y' + p(x)y = r(x)$ where p, r are continuous. The method applies only in case this is possible.
Find Q	Find a simplified formula for $Q = e^{\int p(x)dx}$. The antiderivative $\int p(x)dx$ can be chosen conveniently.
Prepare for Quadrature	Obtain the new equation $\dfrac{(Qy)'}{Q} = r$ by replacing the left side of $y' + p(x)y = r(x)$ by equivalence (4).
Method of Quadrature	Clear fractions to obtain $(Qy)' = rQ$. Apply the method of quadrature to get $Qy = \int r(x)Q(x)dx + C$. Divide by Q to isolate the explicit solution $y(x)$.

In identity (4), functions p, Y and Y' are assumed continuous with p and Y *arbitrary* functions. The integral $\int p(x)dx$ equals $P(x)+C$, where $P(x)$ is some anti-derivative of $p(x)$. Because $e^{\int p(x)dx} = e^{P(x)}e^{C}$, then factor e^C divides out of the fraction in (4). Applications therefore simplify the **integrating factor** $e^{\int p(x)dx}$ to $e^{P(x)}$, where $P(x)$ is *any suitable antiderivative* of $p(x)$ (effectively, we take $C = 0$).

Equation (4) is central to the method, because it collapses the two terms $y' + py$ into a single term $(Qy)'/Q$; the method of quadrature applies to $(Qy)' = rQ$. The literature calls the exponential factor Q an **integrating factor** and equivalence (4) a **factorization** of $Y' + p(x)Y$.

Simplifying an integrating factor. Factor Q is simplified by dropping constants of integration. To illustrate, if $p(x) = 1/x$, then $\int p(x)dx = \ln|x| + C$. The algebra rule $e^{A+B} = e^A e^B$ implies that $Q = e^C e^{\ln|x|} = |x|e^C = (\pm e^C)x$. Let $c_1 = \pm e^C$. Then $Q = c_1 Q_1$ where $Q_1 = x$. The fraction $(Qy)'/Q$ reduces to $(Q_1 y)'/Q_1$, because c_1 cancels. In an application, we choose the simpler expression Q_1. The illustration also shows that the exponential in Q can sometimes be eliminated.

Superposition

Formula (3) can be decomposed into two expressions, called y_h and y_p, so that the **general solution** is expressed as $y = y_h + y_p$. The function y_h solves the homogeneous equation $y' + p(x)y = 0$ and y_p solves the non-homogeneous equation $y' + p(x)y = r(x)$. This observation is called the **superposition principle**.

Equation (3) implies the **homogeneous solution** y_h and a **particular solution** y_p^* can be defined by

$$(5) \qquad y_h = y_0 e^{-\int_{x_0}^{x} p(s)ds}, \quad y_p^* = e^{-\int_{x_0}^{x} p(s)ds} \int_{x_0}^{x} r(t) e^{\int_{x_0}^{t} p(s)ds} dt.$$

Verification amounts to setting $r = 0$ in (3) to determine y_h. The solution y_p^* depends on the forcing term $r(x)$, but y_h does not. Initial conditions of a problem are buried in y_h. Experimentalists view the computation of y_p^* as a *single experiment* in which the state y_p^* is determined by the forcing term $r(x)$ and zero initial data $y = 0$ at $x = x_0$.

Structure of Solutions. Formula (3), proved on page 78, directly establishes existence for the solution to the linear initial value problem (2). The proof also determines what other particular solutions might be used in the formula for a general solution:

Theorem 2 (Solution Structure)
Assume $p(x)$ and $r(x)$ are continuous on $a < x < b$ and $a < x_0 < b$. Let y_h and y_p^* be defined by equation (5). Let y be a solution of $y' + p(x)y = r(x)$ on $a < x < b$. Then y can be decomposed as $y = y_h + y_p^*$, where $y_0 = y(x_0)$.

In short, a linear equation has the solution structure *homogeneous plus particular*. In particular, two solutions of the non-homogeneous equation differ by some solution y_h of the homogeneous equation.

Variation of Parameters

The particular solution $y_p^*(x)$ given by equation (5) is known in the literature as the **variation of parameters formula** (also: **variation of constants**). Because $\int_x^t f = \int_x^{x_0} f + \int_{x_0}^t f$ and $\int_x^{x_0} f = -\int_{x_0}^x f$, the exponential factors in (5) can be re-written in the more compact form

$$y_p^*(x) = \int_{x_0}^x r(t) e^{\int_x^t p(s)ds}\, dt. \tag{6}$$

Terminology. The name comes from the idea of *varying the parameter* y_0 in the formula $y_h = y_0 \mathcal{Q}(x)$, where $\mathcal{Q}(x) = e^{-P(x)}$, $P(x) = \int_{x_0}^x p(s)ds$. The objective is to obtain a *trial solution* y of $y' + p(x)y = r(x)$ in the form $y = y_0(x)\mathcal{Q}(x)$. The unknown $y_0(x)$ is to be determined. History of the derivation appears on page 79.

The Method of Undetermined Coefficients

The method applies to $y' + p(x)y = r(x)$. It finds a particular solution y_p *without* the integration steps present in variation of parameters. The requirements and limitations:

1. Coefficient $p(x)$ of $y' + p(x)y = r(x)$ is constant.
2. The function $r(x)$ is a sum of constants times atoms.

An **atom** is a term having one of the forms

$$x^m,\ x^m e^{ax},\ x^m \cos bx,\ x^m \sin bx,\ x^m e^{ax} \cos bx \quad \text{or} \quad x^m e^{ax} \sin bx.$$

The symbols a and b are real constants, with $b > 0$. Symbol $m \geq 0$ is an integer. The terms x^3, $x \cos 2x$, $\sin x$, e^{-x}, $x^6 e^{-\pi x}$ are atoms. Conversely, given $r(x) = 4\sin x + 5xe^x$, then the atoms of $r(x)$ are $\sin x$ and xe^x.

The Method.

1. Repeatedly differentiate the atoms of $r(x)$ until no new atoms appear. Multiply the distinct atoms so found by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define a **trial solution** y.
2. **Fixup rule**: if solution e^{-px} of $y' + py = 0$ appears in trial solution y, then replace in y matching atoms e^{-px}, xe^{-px}, …by xe^{-px}, x^2e^{-px}, …(other atoms appearing in y are unchanged). The modified expression y is called the **corrected trial solution**.
3. Substitute y into the differential equation $y' + py = r(x)$. Match atoms left and right to write out linear algebraic equations for the undetermined coefficients $d_1, \ldots, d_k$.
4. Solve the equations. The trial solution y with evaluated coefficients $d_1, \ldots, d_k$ becomes the particular solution y_p.

Undetermined Coefficients Illustrated. We will solve

$$y' + 2y = xe^x + 2x + 1 + 3\sin x.$$

Solution:

Test Applicability. The right side $r(x) = xe^x + 2x + 1 + 3\sin x$ is a sum of terms constructed from the atoms xe^x, x, 1, $\sin x$. The left side is $y' + p(x)y$ with $p(x) = 2$, a constant. Therefore, the method of undetermined coefficients applies to find y_p.

Trial Solution. The atoms of $r(x)$ are subjected to differentiation. The distinct atoms so found are 1, x, e^x, xe^x, $\cos x$, $\sin x$ (drop coefficients to identify new atoms). The solution e^{-2x} of $y' + 2y = 0$ does not appear in the list of atoms, so the fixup rule does not apply. Then the trial solution is the expression

$$y = d_1(1) + d_2(x) + d_3(e^x) + d_4(xe^x) + d_5(\cos x) + d_6(\sin x).$$

Equations. To substitute the trial solution y into $y' + 2y$ requires a formula for y':

$$y' = d_2 + d_3e^x + d_4xe^x + d_4e^x - d_5\sin x + d_6\cos x.$$

Then

$$\begin{aligned} r(x) &= y' + 2y \\ &= d_2 + d_3e^x + d_4xe^x + d_4e^x - d_5\sin x + d_6\cos x \\ &\quad + 2d_1 + 2d_2x + 2d_3e^x + 2d_4xe^x + 2d_5\cos x + 2d_6\sin x \\ &= (d_2 + 2d_1)(1) + 2d_2(x) + (3d_3 + d_4)(e^x) + (3d_4)(xe^x) \\ &\quad + (2d_5 + d_6)(\cos x) + (2d_6 - d_5)(\sin x) \end{aligned}$$

Also, $r(x) \equiv 1 + 2x + xe^x + 3\sin x$. Coefficients of atoms on the left and right must match. For instance, constant term $d_1 + 2d_0$ on the left matches constant term 1 on the right. Writing out the matches gives the equations

$$\begin{aligned} 2d_1 + d_2 &= 1, \\ 2d_2 &= 2, \\ 3d_3 + d_4 &= 0, \\ 3d_4 &= 1, \\ 2d_5 + d_6 &= 0, \\ -d_5 + 2d_6 &= 3. \end{aligned}$$

Solve. The first four equations can be solved by back-substitution to give $d_2 = 1$, $d_1 = 0$, $d_4 = 1/3$, $d_3 = -1/9$. The last two equations are solved by elimination or Cramer's rule to give $d_6 = 6/5$, $d_5 = -3/5$.

Report y_p. The trial solution y with evaluated coefficients $d_1, \ldots, d_6$ becomes

$$y_p(x) = x - \frac{1}{9}e^x + \frac{1}{3}xe^x - \frac{3}{5}\cos x + \frac{6}{5}\sin x.$$

A Fixup Rule Illustration. The equation

$$y' + 3y = 8e^x + 3x^2e^{-3x}$$

can be solved by the method of undetermined coefficients to give the general solution $y = y_h + y_p$ where

$$y_h = ce^{-3x}, \quad y_p = 2e^x + x^3e^{-3x}.$$

Solution: The right side $r(x) = 8e^x + 3x^2e^{-3x}$ is constructed from atoms e^x, x^2e^{-3x}. Repeated differentiation of these atoms identifies the new list of atoms e^x, e^{-3x}, xe^{-3x}, x^2e^{-3x}. The fixup rule applies because the solution e^{-3x} of $y' + 3y = 0$ appears in the list. The atoms of the form x^me^{-3x} are multiplied by x to give the new list of atoms e^x, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}. Readers should take note that atom e^x is unaffected by the fixup rule modification. Then the corrected trial solution is

$$y = d_1e^x + d_2xe^{-3x} + d_3x^2e^{-3x} + d_4x^3e^{-3x}.$$

The trial solution expression y is substituted into $y' + 3y = 2e^x + x^2e^{-3x}$ to give the equation

$$4d_1e^x + d_2e^{-3x} + 2d_3xe^{-3x} + 3d_4x^2e^{-3x} = 8e^x + 3x^2e^{-3x}.$$

Coefficients of atoms on each side of the preceding equation are matched to give the equations

$$\begin{aligned} 4d_1 \qquad\qquad\qquad &= 8, \\ d_2 \qquad\qquad &= 0, \\ 2d_3 \qquad &= 0, \\ 3d_4 &= 3. \end{aligned}$$

Then $d_1 = 2$, $d_2 = d_3 = 0$, $d_4 = 1$ and the particular solution is reported to be $y_p = 2e^x + x^3e^{-3x}$.

11 Example (Integrating Factor Method) Solve the equation $2y' + 6y = e^{-x}$ by the integrating factor method.

Solution: The solution is $y = \frac{1}{4}e^{-x} + ce^{-3x}$. The details apply the method which appears on page 71:

$y' + 3y = 0.5e^{-x}$	Divide by 2 to get the standard form.
$Q = e^{3x}$	Find the integrating factor $Q = e^{\int 3dx}$.
$\dfrac{\left(e^{3x}y\right)'}{e^{3x}} = 0.5e^{-x}$	Prepare the equation for quadrature: page 71.
$e^{3x}y = 0.5\int e^{2x}dx$	Clear fractions and apply quadrature.
$y = 0.5e^{-3x}\left(e^{2x}/2 + c_1\right)$	Evaluate the integral. Divide by Q.
$\quad = \frac{1}{4}e^{-x} + ce^{-3x}$	Final answer, $c = 0.5c_1$.

12 Example (Superposition) Find a particular solution of $y' + 2y = 3e^x$ with fewest terms.

Solution: The answer is $y = e^x$. The first step solves the equation using the integrating factor method, giving $y = e^x + ce^{-2x}$; details below. A particular solution with fewest terms, $y = e^x$, is found by setting $c = 0$. The solution y_p^* of equation (5) has two terms: $y_p^* = e^x - e^{3x_0}e^{-2x}$. The reason for the extra term is the condition $y = 0$ at $x = 0$. The two particular solutions differ by the homogeneous solution y_0e^{-2x} where $y_0 = e^{3x_0}$.

Integrating factor method details:

$y' + 2y = 3e^x$	The standard form.
$Q = e^{2x}$	Find the integrating factor $Q = e^{\int 2dx}$.
$\dfrac{\left(e^{2x}y\right)'}{e^{3x}} = 3e^x$	Integrating factor identity applied on the left.
$e^{2x}y = 3\int e^{3x}dx$	Clear fractions and apply quadrature.
$y = e^{-2x}\left(e^{3x} + c\right)$	Evaluate the integral. Isolate y.
$\quad = e^x + ce^{-2x}$	Solution found.

13 Example (Finding y_h and y_p) Find the homogeneous solution y_h and a particular solution y_p for the equation $2xy' + y = 4x^2$ on $x > 0$.

Solution: The solution by the integrating factor method is $y = 0.8x^2 + cx^{-1/2}$; details below. Then $y_h = cx^{-1/2}$ and $y_p = 0.8x^2$ give $y = y_h + y_p$.

The symbol y_p stands for *any* particular solution. Variation of parameters gives a *different* particular solution $y_p^* = 0.8x^2 - 0.8x_0^{5/2}x^{-1/2}$. It differs from the other particular solution $0.8x^2$ by a homogeneous solution $Kx^{-1/2}$.

Integrating factor method details:

$y' + 0.5y/x = 2x$	Standard form. Divided by $2x$.
$Q = e^{0.5\int dx/x}$	The integrating factor is $Q = e^{\int p}$.
$\quad = e^{0.5\ln x}$	Simplify the integration constant.
$\quad = x^{1/2}$	Used $\ln u^n = n\ln u$. Simplified Q found.
$\dfrac{\left(x^{1/2}y\right)'}{x^{1/2}} = 2x$	Integrating factor identity applied on the left.
$x^{1/2}y = 2\int x^{3/2}dx$	Clear fractions. Apply quadrature.
$y = x^{-1/2}\left(4x^{5/2}/5 + c\right)$	Evaluate the integral. Divide to isolate y.
$\quad = 4x^2 + cx^{-1/2}$	Solution found.

14 Example (Classification) Classify the equation $y' = x + \ln(xe^y)$ as linear or non-linear.

Solution: It's linear, with standard linear form $y' + (-1)y = x + \ln x$. To explain why, the term $\ln(xe^y)$ on the right expands into $\ln x + \ln e^y$, which in turn is $\ln x + y$, using logarithm rules. Because $e^y > 0$, then $\ln(xe^y)$ makes sense for only $x > 0$. Henceforth, assume $x > 0$.

Computer algebra test $f(x, y) = f(x, 0) + f_y(x, 0)y$. Expected is LHS − RHS = 0 after simplification. This example produced $\ln e^y - y$ instead of 0, evidence that limitations may exist.

```
assume(x>0):
f:=(x,y)->x+ln(x*exp(y)):
LHS:=f(x,y):
RHS:=f(x,0)+subs(y=0,diff(f(x,y),y))*y:
simplify(LHS-RHS);
```

If the test *passes*, then $y' = f(x, y)$ becomes $y' = f(x, 0) + f_y(x, 0)y$. This example gives $y' = x + \ln x + y$, which converts to the standard linear form $y' + (-1)y = x + \ln x$.

15 **Example (Undetermined Coefficient Method)** Solve the equation $2y' + 6y = 4xe^{-x} + 4xe^{-3x} + 5\sin x$ by the method of undetermined coefficients, verifying $y = y_h + y_p$ is given by the formula

$$y = ce^{-3x} - \frac{1}{2}e^{-x} + xe^{-x} + x^2e^{-3x} - \frac{1}{4}\cos x + \frac{3}{4}\sin x.$$

Solution: The method applies, because the differential equation $2y' + 6y = 0$ has constant coefficients and the right side $r(x) = 4xe^{-x} + 4xe^{-3x} + 5\sin x$ is constructed from the list of atoms xe^{-x}, xe^{-3x}, $\sin x$.

List of Atoms. Differentiate the atoms xe^{-x}, xe^{-3x}, $\sin x$ to find the new list of atoms e^{-x}, xe^{-x}, e^{-3x}, xe^{-3x}, $\cos x$, $\sin x$. The solution e^{-3x} of $2y' + 6y = 0$ appears in the list: the fixup rule applies. Then e^{-3x}, xe^{-3x} are replaced by xe^{-3x}, x^2e^{-3x} to give the corrected list of atoms e^{-x}, xe^{-x}, xe^{-3x}, x^2e^{-3x}, $\cos x$, $\sin x$. Please note that only two of the six atoms were corrected.

Trial solution. The corrected trial solution is

$$y = d_1e^{-x} + d_2xe^{-x} + d_3xe^{-3x} + d_4x^2e^{-3x} + d_5\cos x + d_6\sin x.$$

Substitute y into $2y' + 6y = r(x)$ to give

$$\begin{aligned} r(x) &= 2y' + 6y \\ &= (4d_1 + 2d_2)e^{-x} + 4d_2xe^{-x} + 2d_3e^{-3x} + 4d_4xe^{-3x} \\ &\quad +(2d_6 + 6d_5)\cos x + (6d_6 - 2d_5)\sin x. \end{aligned}$$

Equations. Matching atoms on the left and right of $2y' + 6y = r(x)$, given $r(x) = 4xe^{-x} + 4xe^{-3x} + 5\sin x$, justifies the following equations for the undetermined coefficients; the solution is $d_2 = 1$, $d_1 = -1/2$, $d_3 = 0$, $d_4 = 1$, $d_6 = 3/4$, $d_5 = -1/4$.

$$\begin{aligned} 4d_1 + 2d_2 &= 0, \\ 4d_2 &= 4, \\ 2d_3 &= 0, \\ 4d_4 &= 4, \\ 6d_5 + 2d_6 &= 0, \\ -2d_5 + 6d_6 &= 5. \end{aligned}$$

Report. The trial solution upon substitution of the values for the undetermined coefficients becomes

$$y_p = -\frac{1}{2}e^{-x} + xe^{-x} + x^2e^{-3x} - \frac{1}{4}\cos x + \frac{3}{4}\sin x.$$

Justification of Formula (3): Define

$$\mathcal{Q}(x) = e^{-\int_{x_0}^{x} p(s)ds}, \quad \mathcal{R}(x) = \int_{x_0}^{x} \frac{r(t)}{\mathcal{Q}(t)}dt.$$

The calculus rule $(e^u)' = u'e^u$ and the fundamental theorem of calculus result $(\int_{x_0}^{x} G(t)dt)' = G(x)$ can be used to obtain the formulas

$$\mathcal{Q}' = (-p)\mathcal{Q}, \quad \mathcal{R}' = \frac{r}{\mathcal{Q}}.$$

Existence. Equation (3) is $y = \mathcal{Q}(y_0 + \mathcal{R})$. Existence will be established by showing that y satisfies $y' + py = r$, $y(x_0) = y_0$. The initial condition $y(x_0) = y_0$ follows from $\mathcal{Q}(x_0) = 1$ and $\mathcal{R}(x_0) = 0$. The steps below verify $y' + py = r$, completing existence.

$y' = [\mathcal{Q}(y_0 + \mathcal{R})]'$	Equation (3), using notation $\mathcal{Q}$ and $\mathcal{R}$.
$= \mathcal{Q}'(y_0 + \mathcal{R}) + \mathcal{Q}\mathcal{R}'$	Sum and product rules applied.
$= -p\mathcal{Q}(y_0 + \mathcal{R}) + \mathcal{Q}\mathcal{R}'$	Used $\mathcal{Q}' = (-p)\mathcal{Q}$.
$= -p\mathcal{Q}(y_0 + \mathcal{R}) + r$	Used $\mathcal{R}' = r/\mathcal{Q}$.
$= -py + r$	Apply $y = \mathcal{Q}(y_0 + \mathcal{R})$.

Uniqueness. It remains to show that the solution given by (3) is the only solution. Start by assuming Y is another, subtract them to obtain $u = y - Y$. Then $u' + pu = 0$, $u(x_0) = 0$. To show $y \equiv Y$, it suffices to show $u \equiv 0$.

According to the integrating factor method, the equation $u' + pu = 0$ is equivalent to $(Qu)' = 0$ where $Q = e^{\mathbf{P}}$ and $\mathbf{P}(x) = \int_{x_0}^{x} p(t)dt$. Integrate $(Qu)' = 0$ from x_0 to x, giving $Q(x)u(x) = Q(x_0)u(x_0)$. Since $u(x_0) = 0$ and $Q(x) \neq 0$, it follows that $u(x) = 0$ for all x. This completes the proof.

Remarks on Picard's Theorem. The Picard-Lindelöf theorem, page 53, implies existence-uniqueness, but only on a smaller interval, and furthermore it supplies no practical formula for the solution. Formula (3) is therefore an improvement over the results obtainable from the general theory.

Justification of Factorization (4): It is assumed that $Y(x)$ is a given but otherwise arbitrary differentiable function. Equation (4) will be justified in its fraction-free form

(7) $$\left(e^{\mathbf{P}}Y\right)' = e^{\mathbf{P}}\left(Y' + pY\right), \quad \mathbf{P}(x) = \int p(x)dx.$$

LHS $= \left(e^{\mathbf{P}}Y\right)'$	The left side of equation (7).
$= \left(e^{\mathbf{P}}\right)'Y + e^{\mathbf{P}}Y'$	Apply the product rule $(uv)' = u'v + uv'$.
$= e^{\mathbf{P}}pY + e^{\mathbf{P}}Y'$	Use the chain rule $(e^u)' = e^u u'$ and $\mathbf{P}' = p$.
$= e^{\mathbf{P}}\left(Y' + pY\right)$	The common factor is $e^{\mathbf{P}}$.
= RHS	The right hand side of equation (7).

Historical Account of Variation of Parameters. The *constant* y_0 appearing in y_h is varied to an *unknown function* $y_0(x)$, to be determined subject to the formulas $y = y_0(x)\mathcal{Q}(x)$ and $\mathcal{Q}(x) = e^{-\int_{x_0}^x p(s)ds}$.

$r = y' + py$	The trial solution y will solve $y' + py = r$.
$= (y_0\mathcal{Q})' + py_0\mathcal{Q}$	Substitute $y = y_0(x)\mathcal{Q}(x)$ but suppress x.
$= y_0'\mathcal{Q} + y_0\mathcal{Q}' + py_0\mathcal{Q}$	Apply the product rule $(uv)' = u'v + uv'$.
$= y_0'\mathcal{Q} - y_0 p\mathcal{Q} + py_0\mathcal{Q}$	Apply $\mathcal{Q}' = -p\mathcal{Q}$.
$= y_0'\mathcal{Q}$	Therefore, $y_0'(x) = r(x)/\mathcal{Q}(x)$.

The method of quadrature applies to find $y_0(x) = \int_{x_0}^x r(t)dt/\mathcal{Q}(t)$, because $y_0 = 0$ at $x = x_0$. Then $y = y_0\mathcal{Q}$ duplicates the formula for y_p^* given in (5), which is equivalent to (6).

Exercises 2.3

Integrating Factor Method. Apply the integrating factor method, page 71, to solve the given linear equation. See the examples starting on page 75 for details.

1. $y' + y = e^{-x}$

2. $y' + y = e^{-2x}$

3. $2y' + y = e^{-x}$

4. $2y' + y = e^{-2x}$

5. $2y' + y = 1$

6. $3y' + 2y = 2$

7. $2xy' + y = x$

8. $3xy' + y = 3x$

9. $y' + 2y = e^{2x}$

10. $2y' + y = 2e^{x/2}$

Superposition. Find a particular solution with fewest terms. See Example 12, page 75.

11. $3y' = x$

12. $3y' = 2x$

13. $y' + y = 1$

14. $y' + 2y = 2$

15. $2y' + y = 1$

16. $3y' + 2y = 1$

17. $y' - y = e^x$

18. $y' - y = xe^x$

19. $xy' + y = \sin x$ $(x > 0)$

20. $xy' + y = \cos x$ $(x > 0)$

21. $y' + y = x - x^2$

22. $y' + y = x + x^2$

General Solution. Find y_h and a particular solution y_p. Report the general solution $y = y_h + y_p$. See Example 13, page 76.

23. $y' + y = 1$

24. $xy' + y = 2$

25. $y' + y = x$

26. $xy' + y = 2x$

27. $y' - y = x + 1$

28. $xy' - y = 2x - 1$

29. $2xy' + y = 2x^2$ $(x > 0)$

30. $xy' + y = 2x^2$ $(x > 0)$

Classification. Classify as linear or non-linear. Use the test $f(x,y) = f(x,0) + f_y(x,0)y$ and a computer algebra system, when available, to check the answer. See Example 14, page 76.

31. $y' = 1 + 2y^2$

32. $y' = 1 + 2y^3$

33. $yy' = (1+x)\ln e^y$

34. $yy' = (1+x)(\ln e^y)^2$

35. $y'\sec^2 y = 1 + \tan^2 y$

36. $y' = \cos^2(xy) + \sin^2(xy)$

37. $y'(1+y) = xy$

38. $y' = y(1+y)$

39. $xy' = (x+1)y - xe^{\ln y}$

40. $2xy' = (2x+1)y - xye^{-\ln y}$

Variation of Parameters. Compute the particular solution given by the formula

$$y_p^*(x) = \int_{x_0}^x r(t)e^{\int_x^t p(u)du}dt.$$

41. $y' = x + 1$, $x_0 = 0$

42. $y' = 2x - 1$, $x_0 = 0$

43. $y' + y = e^{-x}$, $x_0 = 0$

44. $y' + y = e^{-2x}$, $x_0 = 0$

45. $y' - 2y = 1$, $x_0 = 0$

46. $y' - y = 1$, $x_0 = 0$

47. $2y' + y = e^x$, $x_0 = 0$

48. $2y' + y = e^{-x}$, $x_0 = 0$

49. $xy' = x + 1$, $x_0 = 1$

50. $xy' = 1 - x^2$, $x_0 = 1$

Undetermined Coefficients. Compute a particular solution y_p according to the method of undetermined coefficients. Report (1) the initial trial solution, (2) the corrected trial solution, (3) the system of equations for the undetermined coefficients and finally (4) the formula for y_p.

51. $y' + y = x + 1$

52. $y' + y = 2x - 1$

53. $y' - y = e^x + e^{-x}$

54. $y' - y = xe^x + e^{-x}$

55. $y' - 2y = 1 + x + e^{2x} + \sin x$

56. $y' - 2y = 1 + x + xe^{2x} + \cos x$

57. $y' + 2y = xe^{-2x} + x^3$

58. $y' + 2y = (2+x)e^{-2x} + xe^x$

59. $y' = x^2 + 4 + xe^x(3 + \cos x)$

60. $y' = x^2 + 5 + xe^x(2 + \sin x)$