Final Exam Differential Equations 2280

Tuesday, 30 April 2019, 7:30am to 10:15am

Instructions: No calculators, notes, tables or books. No answer check is expected. A correct answer without details counts 25%.

Chapters 1 and 2: First Order Differential Equations

Definitions. An **equilibrium solution** is a constant solution, found by replacing all derivatives by zero, then solve for y. If y found by this method is not constant, then the method fails. For $y' + py = q$, the **homogeneous equation** is $y' + py = 0$. An equation $y' = f(x, y)$ is **separable** provided functions F, G exist such that $f(x, y) = F(x)G(y)$.

(a) [20%] Apply a test to the equation $y' = x + y + y^2$, showing it fails to be separable.

(b) [30%] The problem $x\dfrac{dy}{dx} = xy + 3x + 2y + 6$ is both linear and separable. It can be solved by superposition $y = y_h + y_p$, where y_h is the homogeneous solution and y_p is an equilibrium solution. Show details for the answers $y_h = c\,x^2\,e^x$ and $y_p = -3$.

(c) [20%] Solve the linear homogeneous equation $x^2 \dfrac{dy}{dx} + 2y = xy$.

(d) [30%] Solve $2\,\dfrac{d}{dt}v(t) = 5\,e^{-t} + \dfrac{1}{2}\,v(t),\ v(0) = 0$ by the linear integrating factor method. Show all steps.

Chapter 3: Linear Equations of Higher Order

(a) [20%] Solve for the general solution: $y'' + 6y' + 73y = 730$

(b) [30%] Given $5x''(t) + 2x'(t) + 10x(t) = F_0 \cos(\omega t)$, which represents a damped forced spring-mass system with $m = 5$, $c = 2$, $k = 10$, answer questions **(c1), (c2)**.

(c1) Compute the frequency ω for practical mechanical resonance.

(c2) Classify the homogeneous problem as over-damped (non-oscillatory), critically-damped or under-damped (oscillatory).

(c) [20%] Define $y(x) = x\cos(3x) + 3x^3e^x$. Construct the characteristic equation of a linear nth order homogeneous differential equation of least order n which has $y(x)$ as a particular solution.

(d) [30%] An nth order non-homogeneous differential equation is specified by its characteristic equation $(r+1)^3(r^2+100) = 0$ and the forcing term $f(x) = x^2 + x^2\,e^{-x} + x\,e^x + \sin(10x)$. Find the shortest trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** undetermined coefficients.

Chapters 4 and 5: Systems of Differential Equations

(a) [20%] Let $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 5 \end{pmatrix}$, $\vec{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{v}}_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

The eigenpairs of A are $(-2, \vec{\mathbf{v}}_1), (2, \vec{\mathbf{v}}_2), (5, \vec{\mathbf{v}}_3)$.

(a1) Apply the Eigenanalysis Method to solve $\frac{d}{dt}\vec{\mathbf{x}}(t) = A\vec{\mathbf{x}}(t)$.

(a2) Show details for computing eigenpair $(2, \vec{\mathbf{v}}_2)$.

Expected: Show linear algebra details for computing $\vec{\mathbf{v}}_2$ for eigenvalue $\lambda = 2$. This involves row reduction plus display of the scalar solution and the vector solution.

(b) [20%] Find the scalar general solution of the 2×2 system

$$\begin{cases} x' &= 7x + 2y, \\ y' &= 2x + 7y \end{cases} \quad \text{or} \quad \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

by the Cayley-Hamilton-Ziebur Method, using the textbook's Chapter 4 shortcut.

(c) [30%] Assume a 3×3 system $\frac{d}{dt}\vec{\mathbf{u}} = A\vec{\mathbf{u}}$ has a vector general solution

$$\vec{\mathbf{u}}(t) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -c_1 e^{5t} + c_2 e^{8t} \\ c_1 e^{5t} + c_2 e^{8t} \\ c_3 e^{t} \end{pmatrix}.$$

(c1) Compute a 3×3 fundamental matrix $\Phi(t)$.

(c2) Write a formula for the exponential matrix e^{At} as an explicit matrix product. Do not multiply or simplify the product.

(c3) Let $\vec{\mathbf{c}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. Display an explicit matrix-vector product for the solution $\vec{\mathbf{u}}(t)$ of the initial value problem $\frac{d}{dt}\vec{\mathbf{u}} = A\vec{\mathbf{u}}$, $\vec{\mathbf{u}}(0) = \vec{\mathbf{c}}$. Do not multiply or simplify the product.

(d) [30%] Consider the 3×3 linear homogeneous system

$$\begin{cases} x' &= 6x - 2y \\ y' &= -2x + 6y, \\ z' &= y + z \end{cases} \qquad \text{or} \qquad \frac{d}{dt}\vec{\mathbf{u}}(t) = \begin{pmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 1 & 1 \end{pmatrix} \vec{\mathbf{u}}(t).$$

Solve the system by the most efficient method.

Chapter 6: Dynamical Systems

Consider the nonlinear dynamical system

$$(1) \qquad \begin{cases} x' &= 16x - 4x^2 - xy, \\ y' &= 7y - y^2 - xy \end{cases}$$

(a) [20%] Find the four equilibrium points for nonlinear system (1). One of these is $x = 3, y = 4$.

(b) [20%] Compute the Jacobian matrix $J(x, y)$ for nonlinear system (1). Then evaluate $J(x, y)$ at equilibrium point $x = 3, y = 4$.

(c) [30%] Consider nonlinear system (1). Classify the linearization at equilibrium point $x = 3, y = 4$ as a node, spiral, center, saddle. Do not sub-classify a node.

(d) [30%] Consider nonlinear system (1). Determine the possible classification of node, spiral, center or saddle and corresponding stability for equilibrium $x = 3, y = 4$ according to the **Pasting Theorem**, which is Theorem 2 in section 6.2 (Stability of Almost Linear Systems). State precisely the **two exceptions** of the pasting theorem, then explain how the theorem applies to nonlinear system (1) at $x = 3, y = 4$.

Chapter 7: Laplace Theory

Symbol $\delta(t)$ is the Dirac impulse. Symbol $u(t)$ is the unit step. Assumed below is experience with the following. Rules have precise hypotheses, omitted here for brevity.

Convolution Theorem. $\mathcal{L}(g_1)\mathcal{L}(g_2) = \mathcal{L}\left(\int_0^t g_1(t-x)g_2(x)dx\right)$

Periodic Function Theorem. $f(t+p) = f(t)$ implies $\mathcal{L}(f(t)) = \dfrac{\int_0^p f(t)dt}{1-e^{-ps}}$

Second Shifting Theorem Forward. $\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(t)|_{t->t+a}\right)$

Second Shifting Theorem Backward. $e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)u(t-a))$

Dirac Impulse Identity. $\int_0^\infty W(x)du(t-a) = W(a)$. Formally $\delta(t) = du(t)$. Then $\mathcal{L}(\delta(t-a)) = e^{-as}$ for $a \ge 0$.

Resolvent Identity. $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$ has identity $(sI - A)\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}(0) + \mathcal{L}(\vec{\mathbf{F}})$.

Exponential Order. This means $|f(x)| \le M\, e^{\alpha x}$ for some $M > 0$ and real number α.

(a) [20%] Let $f(t)$ be continuous and of exponential order. Define $F(s) = \mathcal{L}(f(t))$. Prove the **Final Value Theorem**: $\lim_{s\to\infty} F(s) = 0$ (succinctly $F(\infty) = 0$).

(b) [20%] Illustrate the convolution theorem by solving for $f(t)$ in the equation $\mathcal{L}(f(t)) = \frac{1}{s}\frac{1}{s+1}$. Check the answer with partial fractions.

(c) [20%] Solve for $f(t)$ using the second shifting theorem: $\mathcal{L}(f(t)) = e^{-2s}\frac{1}{s+1}$.

(d) [20%] Symbol $\delta(t)$ is the Dirac impulse. Derive an expression for $\mathcal{L}(x(t))$ for the impulse problem

$$x''(t) + 100x(t) = 5\delta(t - \pi), \quad x(0) = 0, \quad x'(0) = 1.$$

To save time, do not solve for $x(t)$.

(e) [20%] **Laplace Theory** applied to the forced linear dynamical system

$$\begin{cases} x' = \quad 4x - 2y + 2t, \\ y' = -2x + 4y, \\ x(0) = 0, y(0) = 0, \end{cases} \qquad \text{or} \qquad \begin{cases} \vec{\mathbf{u}}' = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} \vec{\mathbf{u}} + \begin{pmatrix} 2t \\ 0 \end{pmatrix}, \\ \vec{\mathbf{u}}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases}$$

produces the formulas

$$\mathcal{L}(x(t)) = \frac{2s - 8}{s^2(s-2)(s-6)}, \quad \mathcal{L}(y(t)) = \frac{-4}{s^2(s-2)(s-6)}.$$

Display the **Resolvent Method** solution steps that produce these formulas. To save time, **do not solve for** $x(t)$ **or** $y(t)$.

Chapter 9: Fourier Series and Partial Differential Equations

In part (a), let $f_0(x) = 2$ on the interval $1 < x < 2$, $f_0(x) = 0$ for all other values of x on $-2 \le x \le 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4. The Fourier series of $f(x)$ on $|x| \le L$ is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L).$$

Formulas exist for a_n, b_n expressed in terms of f, using inner product spaces.

(a) [20%] Compute the Fourier coefficients a_5 and b_5 of $f(x)$ on $[-2, 2]$. Warning: f is neither even nor odd.

In part (b), let $g_0(x) = 1$ on the interval $-2 < x < 0$, $g_0(x) = 2$ on the interval $0 < x < 2$, $g_0(x) = 0$ for all other values of x on $-2 \leq x \leq 2$. Let $g(x)$ be the periodic extension of g_0 to the whole real line, of period 4.

(b) [10%] Find all values of x in $-3 < x < 5$ for which the Fourier series of g will exhibit Gibb's over-shoot.

(c) [10%] Assume $h(x)$ is a piecewise continuous function on $(-\infty, \infty)$. Let $H(x)$ be the Fourier series of $h(x)$ on $-L \leq x \leq L$. Does $H(0) = h(0)$ hold no matter the choice of h? Cite a theorem or invent a counterexample.

(d) [30%] **Heat Conduction in a Rod**.
Let $L = 2$ (rod length), $k = 1$ (conduction constant). Solve the rod problem on $0 \le x \le L$, $t \ge 0$:

$$\begin{cases} u_t & = & k\, u_{xx}, \\ u(0,t) & = & 0, \\ u(L,t) & = & 0, \\ u(x,0) & = & \displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1} \sin(2n\pi x) \end{cases}$$

DEFINITION. The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ on $0 < x < L$, $t > 0$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

Each normal mode is a solution of the string equation. A superposition of normal modes is also a solution of the string equation.

(e) [30%] **Vibration of a Finite String**.
Let $L = 4$ (string length), $c = 4$ (wave speed). Solve the finite string vibration problem on $0 \le x \le L$, $t > 0$:

$$\begin{cases} u_{tt}(x,t) &= c^2\, u_{xx}(x,t), \\ u(0,t) &= 0, \\ u(4,t) &= 0, \\ u(x,0) &= \sin(5\pi x) + 5\sin(7\pi x), \\ u_t(x,0) &= \sin(7\pi x) + 12\sin(15\pi x). \end{cases}$$
