Final Exam Differential Equations 2280

Monday, 30 April 2018, 12:45pm-3:15pm

Instructions: Time limit 120 minutes. No calculators, notes, tables or books. No answer check is expected. A correct answer without details counts 25%.

Chapters 1 and 2: Linear First Order Differential Equations

(a) [40%] Solve $7\frac{d}{dt}v(t) = 11 + \frac{6}{t+1}v(t)$, $v(0) = 11$. Show all integrating factor steps.

(b) [30%] Solve the linear homogeneous equation $x^2\frac{dy}{dx} = y + xy$.

(c) [30%] The problem $x^2\frac{dy}{dx} = y + 2x + 2 + xy$ is both linear and separable. It can be solved by linear theory using superposition $y = y_h + y_p$, where y_p is an equilibrium solution. Find y_h and y_p.

Chapter 3: Linear Equations of Higher Order

(a) [10%] Solve for the general solution: $y'' - 6y' + 25y = 0$

(b) [20%] Solve for the general solution: $y^{(5)} - 6y^{(4)} + 25y^{(3)} = 0$

(c) [20%] An nth order linear homogeneous differential equation has characteristic equation $r(r^3 - r)^2(r^2 - 6r + 25)^2 = 0$. Solve for the general solution.

(d) [20%] Construct the characteristic equation of a linear nth order homogeneous differential equation of least order n which has a particular solution $y(x) = x\cos(2x) + 3x^4e^x + e^x\sin(3x)$.

(e) [30%] An nth order non-homogeneous differential equation is specified by its characteristic equation $r^2(r+1)^3(r^2+16) = 0$ and the forcing term $f(x) = x^2 + x^3e^{-x} + xe^{3x} + \sin(4x)$. Find the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate undetermined coefficients.

Chapters 4 and 5: Systems of Differential Equations

(a) [10%] Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 4 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Find all eigenpairs of A and then write the solution of $\vec{\mathbf{x}}(t)$ of $\frac{d}{dt}\vec{\mathbf{x}}(t) = A\vec{\mathbf{x}}(t)$ according to the Eigenanalysis Method.

(b) [20%] Find the general solution of the 2×2 system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

according to the Cayley-Hamilton-Ziebur Method, using the textbook's Chapter 4 shortcut.

(c) [10%] Assume a 3×3 system $\frac{d}{dt}\vec{\mathbf{u}} = A\vec{\mathbf{u}}$ has a scalar general solution

$$u_1(t) = -c_1e^{4t} + c_2e^{8t}, \quad u_2(t) = c_1e^{4t} + c_2e^{8t}, \quad u_3(t) = c_3e^{t}$$

Compute a 3×3 fundamental matrix $\Phi(t)$ and then write a formula for the exponential matrix e^{At}. Do not simplify any formula.

(d) [20%] Consider the 3×3 linear homogeneous system

$$\begin{cases} x' &= 5x - y \\ y' &= -x + 5y, \\ z' &= x + z \end{cases} \qquad \text{or} \qquad \frac{d}{dt}\vec{\mathbf{u}}(t) = \begin{pmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}\vec{\mathbf{u}}(t).$$

Solve the system by the most efficient method.

Chapter 6: Dynamical Systems

Consider the nonlinear dynamical system

$$\begin{cases} x' &= 16x - 2x^2 - xy, \\ y' &= 14y - 2y^2 - xy \end{cases} \tag{1}$$

(a) [20%] Find the equilibrium points for nonlinear system (1).

(b) [20%] Compute the Jacobian matrix $J(x, y)$ for nonlinear system (1). Then evaluate $J(x, y)$ at each of the equilibrium points found in part (a).

(c) [30%] Consider nonlinear system (1). Classify the linearization at each equilibrium point found in part (a) as a node, spiral, center, saddle. Do not sub-classify a node.

(d) [30%] Consider nonlinear system (1). Determine the possible classifications of node, spiral, center or saddle and corresponding stability for each equilibrium determined in part (a), according to the **Pasting Theorem**, which is Theorem 2 in section 6.2 (Stability of Almost Linear Systems).

Chapter 7: Laplace Theory

(a) [20%] Solve for $f(t)$ in the relation $\mathcal{L}(f) = \left.\left(\frac{d}{ds}\mathcal{L}\left(t^2 e^{5t}\sin t\right)\right)\right|_{s\to s+2}$.

(b) [20%] Find $\mathcal{L}(f)$ given $f(t) = (-t)e^t + t\,e^{-t}\sin(2t)$.

(c) [30%] Consider the forced linear dynamical system

$$\begin{cases} x' &= 5x - y + 2t, \\ y' &= -x + 5y + 1. \end{cases}$$

Show that subject to initial states $x(0) = 0$, $y(0) = 0$ the solution $x(t), y(t)$ satisfies

$$\mathcal{L}(x(t)) = \frac{s-10}{s^2(s-4)(s-6)}, \quad \mathcal{L}(y(t)) = \frac{s^2-5s-2}{s^2(s-4)(s-6)}.$$

(d) [30%] Solve for $x(t)$ in the relation $\mathcal{L}(x(t)) = \frac{s-10}{s^2(s-4)(s-6)}$. Leave the partial fraction constants unevaluated.

Chapter 9: Fourier Series and Partial Differential Equations

In parts (a) and (b), let $f_0(x) = -1$ on the interval $-2 < x < -1$, $f_0(x) = 1$ on the interval $1 < x < 2$, $f_0(x) = 0$ for all other values of x on $-2 \le x \le 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(a) [20%] Compute the Fourier coefficients of $f(x)$ on $[-2, 2]$.

(b) [10%] Find all values of x in $|x| < 4$ which will exhibit Gibb's over-shoot.

(c) [30%] **Heat Conduction in a Rod**. Solve the rod problem on $0 \le x \le 2$, $t \ge 0$:

$$\begin{cases} u_t &= u_{xx}, \\ u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= 2\sin(\pi x) + 5\sin(2\pi x) \end{cases}$$

(d) [40%] **Vibration of a Finite String**. Solve the finite string vibration problem on $0 \le x \le 4$, $t > 0$:

$$\begin{cases} u_{tt}(x,t) &= 16u_{xx}(x,t), \\ u(0,t) &= 0, \\ u(5,t) &= 0, \\ u(x,0) &= \sin(5\pi x) + 2\sin(7\pi x), \\ u_t(x,0) &= \sin(7\pi x) + \sin(10\pi x). \end{cases}$$