Differential Equations 2280

Sample Midterm Exam 3 with Solutions
Exam Date: Friday 13 April 2018 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. Problems below cover the possibilities, but the exam day content will be much less, as was the case for Exams 1, 2.

Chapter 3

1. (Linear Constant Equations of Order n)

(a) Find by variation of parameters a particular solution y_p for the equation $y'' = 1 - x$. Show all steps in variation of parameters. Check the answer by quadrature.

(b) A particular solution of the equation $mx'' + cx' + kx = F_0 \cos(2t)$ happens to be $x(t) = 11\cos 2t + e^{-t}\sin\sqrt{11}t - \sqrt{11}\sin 2t$. Assume m, c, k all positive. Find the unique periodic steady-state solution $x_{\rm SS}$.

(c) A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions $2e^{3x} + 4x$ and xe^{3x}. Write a formula for the general solution.

(d) Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 64x = 40\cos(4t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time, don't convert to phase-amplitude form**.

(e) Write the solution $x(t)$ of

$$x''(t) + 25x(t) = 180\sin(4t), \quad x(0) = x'(0) = 0,$$

as the sum of two harmonic oscillations of different natural frequencies.
To save time, don't convert to phase-amplitude form.

(f) Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 2x = 5\sin(t)$.

(g) Given $5x''(t) + 2x'(t) + 4x(t) = 0$, which represents a damped spring-mass system with $m = 5$, $c = 2$, $k = 4$, determine if the equation is over-damped , critically damped or under-damped.
To save time, do not solve for $x(t)$!

(h) Determine the practical resonance frequency ω for the electric current equation

$$2I'' + 7I' + 50I = 100\omega\cos(\omega t).$$

(i) Given the forced spring-mass system $x'' + 2x' + 17x = 82\sin(5t)$, find the steady-state periodic solution.

(j) Let $f(x) = x^3e^{1.2x} + x^2e^{-x}\sin(x)$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation of least order which has $f(x)$ as a solution. To save time, do not expand the polynomial and do not find the differential equation.

Answers and Solution Details:

Part (a) Answer: $y_p = \dfrac{x^2}{2} - \dfrac{x^3}{6}$.

Variation of Parameters.

Solve $y'' = 0$ to get $y_h = c_1y_1 + c_2y_2$, $y_1 = 1$, $y_2 = x$. Compute the Wronskian $W = y_1y_2' - y_1'y_2 = 1$. Then for $f(t) = 1 - x$,

$y_p = y_1 \int y_2 \frac{-f}{W} dx + y_2 \int y_1 \frac{f}{W} dx,$

$y_p = 1 \int -x(1-x)dx + x \int 1(1-x)dx,$

$y_p = -1(x^2/2 - x^3/3) + x(x - x^2/2),$

$y_p = x^2/2 - x^3/6.$

This answer is checked by quadrature, applied twice to $y'' = 1 - x$ with initial conditions zero.

Part (b) It has to be the terms left over after striking out the transient terms, those terms with limit zero at infinity. Then $x_{SS}(t) = 11\cos 2t - \sqrt{11}\sin 2t$.

Part (c) In order for xe^{3x} to be a solution, the general solution must have Euler atoms e^{3x}, xe^{3x}. Then the first solution $2e^{3x} + 4x$ minus 2 times the Euler atom e^{3x} must be a solution, therefore x is a solution, resulting in Euler atoms $1, x$. The general solution is then a linear combination of the four Euler atoms: $y = c_1(1) + c_2(x) + c_3\left(e^{3x}\right) + c_4\left(xe^{3x}\right)$.

Part (d) Use undetermined coefficients trial solution $x = d_1\cos 4t + d_2 \sin 4t$. Then $d_1 = 5/6$, $d_2 = 0$, and finally $x_p(t) = (5/6)\cos(4t)$. The characteristic equation $r^2 + 64 = 0$ has roots $\pm 8i$ with corresponding Euler solution atoms $\cos(8t), \sin(8t)$. Then $x_h(t) = c_1\cos(8t) + c_2\sin(8t)$. The general solution is $x = x_h + x_p$. Now use $x(0) = x'(0) = 0$ to determine $c_1 = -5/6, c_2 = 0$, which implies the particular solution $x(t) = -\frac{5}{6}\cos(8t) + \frac{5}{6}\cos(4t)$.

Part (e) The answer is $x(t) = -16\sin(5t) + 20\sin(4t)$ by the method of undetermined coefficients. Rule I: $x = d_1\cos(4t) + d_2\sin(4t)$. Rule II does not apply due to natural frequency $\sqrt{25} = 5$ not equal to the frequency of the trial solution (no conflict). Substitute the trial solution into $x''(t) + 25x(t) = 180\sin(4t)$ to get $9d_1\cos(4t) + 9d_2\sin(4t) = 180\sin(4t)$. Match coefficients, to arrive at the equations $9d_1 = 0$, $9d_2 = 180$. Then $d_1 = 0$, $d_2 = 20$ and $x_p(t) = 20\sin(4t)$. Lastly, add the homogeneous solution to obtain $x(t) = x_h + x_p = c_1\cos(5t) + c_2\sin(5t) + 20\sin(4t)$. Use the initial condition relations $x(0) = 0, x'(0) = 0$ to obtain the equations $\cos(0)c_1 + \sin(0)c_2 + 20\sin(0) = 0$, $-5\sin(0)c_1 + 5\cos(0)c_2 + 80\cos(0) = 0$. Solve for the coefficients $c_1 = 0$, $c_2 = -16$

Part (f) The answer is $x = \sin t - 2\cos t$ by the method of undetermined coefficients. Rule I: the trial solution $x(t)$ is a linear combination of the Euler atoms found in $f(x) = 5\sin(t)$. Then $x(t) = d_1\cos(t) + d_2\sin(t)$. Rule II does not apply, because solutions of the homogeneous problem contain negative exponential factors (no conflict). Substitute the trial solution into $x'' + 2x' + 2x = 5\sin(t)$ to get $(-2d_1 + d_2)\sin(t) + (d_1 + 2d_2)\cos(t) = 5\sin(t)$. Match coefficients to find the system of equations $(-2d_1 + d_2) = 5$, $(d_1 + 2d_2) = 0$. Solve for the coefficients $d_1 = -2$, $d_2 = 1$.

Part (g) Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is $b^2 - 4ac = 2^2 - 4(5)(4) = (19)(-4)$, therefore there are two complex conjugate roots and the equation is **under-damped**. Alternatively, factor $5r^2 + 2r + 4$ to obtain roots $(-1 \pm \sqrt{19}i)/5$ and then classify as **under-damped**.

Part (h) The resonant frequency is $\omega = 1/\sqrt{LC} = 1/\sqrt{2/50} = \sqrt{25} = 5$. The solution uses the theory in the textbook, section 3.7, which says that electrical resonance occurs for $\omega = 1/\sqrt{LC}$.

Part (i) The answer is $x(t) = -5\cos(5t) - 4\sin(5t)$ by undetermined coefficients.
Rule I: The trial solution is $x_p(t) = A\cos(5t) + B\sin(5t)$. Rule II: because the homogeneous solution $x_h(t)$ has limit zero at $t = \infty$, then Rule II does not apply (no conflict). Substitute the trial solution into the differential equation. Then $-8A\cos(5t) - 8B\sin(5t) - 10A\sin(5t) + 10B\cos(5t) = 82\sin(5t)$. Matching coefficients of sine and cosine gives the equations $-8A + 10B = 0$, $-10A - 8B = 82$. Solving, $A = -5, B = -4$. Then $x_p(t) = -5\cos(5t) - 4\sin(5t)$ is the unique periodic steady-state solution.

Part (j) The characteristic polynomial is the expansion $(r - 1.2)^4((r + 1)^2 + 1)^3$. Because x^3e^{ax} is an Euler solution atom for the differential equation if and only if $e^{ax}, xe^{ax}, x^2e^{ax}, x^3e^{ax}$ are Euler solution atoms, then the characteristic equation must have roots $1.2, 1.2, 1.2, 1.2$, listing according to multiplicity. Similarly, $x^2e^{-x}\sin(x)$ is an Euler solution atom for the differential equation if and only if $-1\pm i, -1\pm i, -1\pm i$ are roots of the characteristic equation. There is a total of 10 roots with product of the factors $(r-1)^4((r+1)^2+1)^3$ equal to the 10th degree characteristic polynomial.

Chapters 4 and 5

2. **(Systems of Differential Equations)**

Background. Let A be a real 3×3 matrix with eigenpairs $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $(\lambda_3, \mathbf{v}_3)$. The eigenanalysis method says that the 3×3 system $\mathbf{x}' = A\mathbf{x}$ has general solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + c_3\mathbf{v}_3e^{\lambda_3 t}.$$

Background. Let A be an $n \times n$ real matrix. The method called **Cayley-Hamilton-Ziebur** is based upon the result

> The components of solution $\mathbf{x}$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ are linear combinations of Euler solution atoms obtained from the roots of the characteristic equation $|A - \lambda I| = 0$.

Background. Let A be an $n\times n$ real matrix. An augmented matrix $\Phi(t)$ of n independent solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is called a **fundamental matrix**. It is known that the general solution is $\mathbf{x}(t) = \Phi(t)\mathbf{c}$, where $\mathbf{c}$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, $|\Phi(0)| \neq 0$.

(a) Display eigenanalysis details for the 3×3 matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix},$$

then display the general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(b) The 3×3 triangular matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix},$$

represents a linear cascade, such as found in brine tank models. Using the linear integrating factor method, starting with component $x_3(t)$, find the vector general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(c) The exponential matrix e^{At} is defined to be a fundamental matrix $\Psi(t)$ selected such that $\Psi(0) = I$, the $n\times n$ identity matrix. Justify the formula $e^{At} = \Phi(t)\Phi(0)^{-1}$, valid for *any* fundamental matrix $\Phi(t)$.

(d) Let A denote a 2×2 matrix. Assume $\mathbf{x}'(t) = A\mathbf{x}(t)$ has scalar general solution $x_1 = c_1e^t + c_2e^{2t}$, $x_2 = (c_1 - c_2)e^t + 2c_1 + c_2)e^{2t}$, where c_1, c_2 are arbitrary constants. Find a fundamental matrix $\Phi(t)$ and then go on to find e^{At} from the formula in part (c) above.

(e) Let A denote a 2×2 matrix and consider the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Assume fundamental matrix $\Phi(t) = \begin{pmatrix} e^t & e^{2t} \\ 2e^t & -e^{2t} \end{pmatrix}$. Find the 2×2 matrix A.

(f) The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = 3x + y, \quad y' = -x + 3y,$$

which has complex eigenvalues $\lambda = 3 \pm i$. Show the details of the method, then go on to report a fundamental matrix $\Phi(t)$.

Remark. The vector general solution is $\mathbf{x}(t) = \Phi(t)\mathbf{c}$, which contains no complex numbers. Reference: 4.1, Examples 6,7,8.

Answers and Solution Details:

Part (a) The details should solve the equation $|A - \lambda I| = 0$ for the three eigenvalues $\lambda = 5, 4, 3$. Then solve the three systems $(A - \lambda I)\vec{v} = \vec{0}$ for eigenvector $\vec{v}$, for $\lambda = 5, 4, 3$.
The eigenpairs are

$$5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 4, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Part (b) Write the system in scalar form

$$\begin{aligned} x' &= 3x + y + z, \\ y' &= 4y + z, \\ z' &= 5z. \end{aligned}$$

Solve the last equation as
$z = \dfrac{\text{constant}}{\text{integrating factor}} = c_3 e^{5t}$.
$\boxed{z = c_3 e^{5t}}$
The second equation is
$y' = 4y + c_3 e^{5t}$
The linear integrating factor method applies.
$y' - 4y = c_3 e^{-5t}$
$\dfrac{(Wy)'}{W} = c_3 e^{5t}$, where $W = e^{-4t}$,
$(Wy)' = c_3 W e^{5t}$
$(e^{-4t} y)' = c_3 e^{-4t} e^{5t}$
$e^{-4t} y = c_3 e^t + c_2$.
$\boxed{y = c_3 e^{5t} + c_2 e^{4t}}$
Stuff these two expressions into the first differential equation:
$x' = 3x + y + z = 3x + 2c_3 e^{5t} + c_2 e^{4t}$
Then solve with the linear integrating factor method.
$x' - 3x = 2c_3 e^{5t} + c_2 e^{4t}$
$\dfrac{(Wx)'}{W} = 2c_3 e^{5t} + c_2 e^{4t}$, where $W = e^{-3t}$. Cross-multiply:
$(e^{-3t} x)' = 2c_3 e^{5t} e^{-3t} + c_2 e^{4t} e^{-3t}$, then integrate:
$e^{-3t} x = c_3 e^{2t} + c_2 e^t + c_1$
$e^{-3t} x = c_3 e^{2t} + c_2 e^t + c_1$, divide by e^{-3t}:
$\boxed{x = c_3 e^{5t} + c_2 e^{4t} + c_1 e^{3t}}$

Part (c) The question reduces to showing that e^{At} and $\Phi(t)\Phi(0)^{-1}$ have equal columns. This is done by showing that the matching columns are solutions of $\vec{u}' = A\vec{u}$ with the same initial condition $\vec{u}(0)$, then apply Picard's theorem on uniqueness of initial value problems.

Part (d) Take partial derivatives on the symbols c_1, c_2 to find vector solutions $\vec{v}_1(t)$, $\vec{v}_2(t)$. Define $\Phi(t)$ to be the augmented matrix of $\vec{v}_1(t)$, $\vec{v}_2(t)$. Compute $\Phi(0)^{-1}$, then multiply on the right of $\Phi(t)$ to obtain

$e^{At} = \Phi(t)\Phi(0)^{-1}$. Check the answer in a computer algebra system or using Putzer's formula.

Part (e) The equation $\Phi'(t) = A\Phi(t)$ holds for every t. Choose $t = 0$ and then solve for $A = \Phi'(0)\Phi(0)^{-1}$.

Part (f) By C-H-Z, $x = c_1 e^{3t}\cos(t) + c_2 e^{3t}\sin(t)$. Isolate y from the first differential equation $x' = 3x + y$, obtaining the formula $y = x' - 3x = -c_1 e^{3t}\sin(t) + c_2 e^{3t}\cos(t)$. A fundamental matrix is found by taking partial derivatives on the symbols c_1, c_2. The answer is exactly the Jacobian matrix of $\begin{pmatrix} x \\ y \end{pmatrix}$ with respect to variables c_1, c_2.

$\Phi(t) = \begin{pmatrix} e^{3t}\cos(t) & e^{3t}\sin(t) \\ -e^{3t}\sin(t) & e^{3t}\cos(t) \end{pmatrix}$.

Chapter 6

3. (Linear and Nonlinear Dynamical Systems)

(a) Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \vec{u}$$

(b) Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \vec{u}$$

(c) Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - 2y^2 - y + 32, \\ y' &= 2x^2 - 2xy. \end{aligned}$$

An equilibrium point is $x = 4$, $y = 4$. Compute the Jacobian matrix $A = J(4, 4)$ of the linearized system at this equilibrium point.

(d) Consider the nonlinear dynamical system

$$\begin{aligned} x' &= -x - 2y^2 - y + 32, \\ y' &= 2x^2 + 2xy. \end{aligned}$$

An equilibrium point is $x = -4$, $y = 4$. Compute the Jacobian matrix $A = J(-4, 4)$ of the linearized system at this equilibrium point.

(e) Consider the nonlinear dynamical system $\begin{cases} x' = -4x + 4y + 9 - x^2, \\ y' = 3x - 3y. \end{cases}$

At equilibrium point $x = 3$, $y = 3$, the Jacobian matrix is $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the linear system $\frac{d}{dt}\vec{u} = A\vec{u}$.

(2) Apply the Pasting Theorem to classify $x = 3$, $y = 3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

(f) Consider the nonlinear dynamical system $\begin{cases} x' &= -4x - 4y + 9 - x^2, \\ y' &= 3x + 3y. \end{cases}$

At equilibrium point $x = 3$, $y = -3$, the Jacobian matrix is $A = J(3, -3) = \begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix}$.

Linearization. Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\frac{d}{dt}\vec{u} = A\vec{u}$.

Nonlinear System. Apply the Pasting Theorem to classify $x = 3$, $y = -3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

Answers and Solution Details:

Part (a) It is an unstable spiral. Details: The eigenvalues of A are roots of $r^2-2r+5=(r-1)^2+4=0$, which are complex conjugate roots $1\pm 2i$. Rotation eliminates the saddle and node. Finally, the atoms $e^t\cos 2t$, $e^t\sin 2t$ have limit zero at $t=-\infty$, therefore the system is stable at $t=-\infty$ and unstable at $t=\infty$. So it must be a spiral [centers have no exponentials]. Report: **unstable spiral**.

Part (b) It is a stable spiral. Details: The eigenvalues of A are roots of $r^2+2r+5=(r+1)^2+4=0$, which are complex conjugate roots $-1\pm 2i$. Rotation eliminates the saddle and node. Finally, the atoms $e^{-t}\cos 2t$, $e^{-t}\sin 2t$ have limit zero at $t=\infty$, therefore the system is stable at $t=\infty$ and unstable at $t=-\infty$. So it must be a spiral [centers have no exponentials]. Report: **stable spiral**.

Part (c) The Jacobian is $J(x,y)=\begin{pmatrix} 1 & -4y-1 \\ 4x-2y & -2x \end{pmatrix}$. Then $A=J(4,4)=\begin{pmatrix} 1 & -17 \\ 8 & -8 \end{pmatrix}$.

Part (d) The Jacobian is $J(x,y)=\begin{pmatrix} -1 & -4y-1 \\ 4x+2y & 2x \end{pmatrix}$. Then $A=J(-4,4)=\begin{pmatrix} -1 & -17 \\ -8 & -8 \end{pmatrix}$.

Part (e) (1) The Jacobian is $J(x,y)=\begin{pmatrix} -4-2x & 4 \\ 3 & -3 \end{pmatrix}$. Then $A=J(3,3)=\begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$. The eigenvalues of A are found from $r^2+13r+18=0$, giving distinct real negative roots $-\frac{13}{2}\pm(\frac{1}{2})\sqrt{97}$. Because there are no trig functions in the Euler solution aistoms, then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms limit to zero at $t=\infty$, therefore it is a node and we report a **stable node** for the linear problem $\vec{u}'=A\vec{u}$ at equilibrium $\vec{u}=\vec{0}$.
(2) Theorem 2 in Edwards-Penney section 6.2 applies to say that the same is true for the nonlinear system: **stable node** at $x=3$, $y=3$. The exceptional case in Theorem 2 is a proper node, having characteristic equation roots that are equal. Stability is always preserved for nodes.

Part (f)

Linearization. The Jacobian is $J(x,y)=\begin{pmatrix} -4-2x & -4 \\ 3 & 3 \end{pmatrix}$. Then $A=J(3,3)=\begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix}$. The eigenvalues of A are found from $r^2+7r-18=0$, giving distinct real roots $2,-9$. Because there are no trig functions in the Euler solution atoms e^{2t},e^{-9t}, then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms do not limit to zero at $t=\infty$ or $t=-\infty$, therefore it is a saddle and we report a **unstable saddle** for the linear problem $\vec{u}'=A\vec{u}$ at equilibrium $\vec{u}=\vec{0}$.
Nonlinear System. Theorem 2 in Edwards-Penney section 6.2 applies to say that the same is true for the nonlinear system: **unstable saddle** at $x=3$, $y=3-$.

Final Exam Problems

Chapter 5. Solve a homogeneous system $u'=Au$, $u(0)=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $A=\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ using the matrix exponential, Zeibur's method, Laplace resolvent and eigenanalysis.

Chapter 5. Solve a non-homogeneous system $u'=Au+F(t)$, $u(0)=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A=\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $F(t)=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ using variation of parameters.