Differential Equations 2280

Midterm Exam 3

Exam Date: 13 April 2018 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Chapter 3 – Nth Order Differential Equations

Problem (1a) [40%] Find the **Beats** solution for a forced undamped spring-mass problem

$$x'' + \sigma^2 x = F_0 \cos(\omega t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time**, please don't convert your answer.

Problem (1b) [30%] Let $f(x)$ be a given linear combination of Euler solution atoms. Find the characteristic equation of a linear homogeneous scalar differential equation of least order such that $y = f(x)$ is a solution. Kindly leave the characteristic equation in factored form, unexpanded.

Problem (1c) [40%] Consider a forced mechanical oscillation equation and/or a forced electrical current equation. Determine the practical resonance frequency ω for each equation. Determine a particular solution by the method of undetermined coefficients. Find the amplitude of this particular solution.

Chapters 4 and 5 – Systems of Differential Equations

Theorem. (Eigenanalysis Method) If A is a real 3×3 matrix with eigenpairs $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$, $(\lambda_3, \vec{v}_3)$, then the system $\vec{x}' = A\vec{x}$ has general solution

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + c_3 \vec{v}_3 e^{\lambda_3 t}.$$

Theorem. (Cayley-Hamilton-Ziebur). The components of solution $\vec{x}$ of $\vec{x}'(t) = A\vec{x}(t)$ are linear combinations of Euler solution atoms obtained from the roots of the characteristic equation $|A - \lambda I| = 0$.

Definition. Let A be an $n \times n$ real matrix. An augmented matrix $\Phi(t)$ of n independent solutions of $\vec{x}'(t) = A\vec{x}(t)$ is called a **fundamental matrix**. It is known that the general solution is $\vec{x}(t) = \Phi(t)\vec{c}$, where $\vec{c}$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, $|\Phi(0)| \neq 0$.

Chapters 4 and 5 – Systems of Differential Equations

Problem (2a) [30%] Assume given a specific 3×3 matrix A with given eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Apply the Cayley-Hamilton-Ziebur theorem to this example.

Problem (2b) [40%] A linear cascade, typically found in brine tank models, satisfies $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ where the 4×4 matrix and vector $\vec{x}$ are defined by

$$A = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Use an appropriate method to find the vector general solution $\vec{x}(t)$ of $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$.

Problem (2c) [40%] A linear cascade, typically found in brine tank models, satisfies $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ where the 4×4 matrix and vector $\vec{x}$ are defined by

$$A = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Apply Laplace's method to obtain a 4×4 system for $\mathcal{L}(x_1), \mathcal{L}(x_2), \mathcal{L}(x_3), \mathcal{L}(x_4)$. Your solution can use scalar equations or the vector-matrix equation $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$. Other parts of this problem: Solve the system using Cramer's Rule. Solve for $\vec{x}$ using Laplace tables and Lerch's theorem.

Problem (2d) [30%] The Cayley-Hamilton-Ziebur shortcut is to be applied to a system

$$x' = ax + by, \quad y' = cx + dy,$$

where a, b, c, d are given along with the eigenvalues λ_1, λ_2.

- **Part 1**. Show the details of the method, finally displaying formulas for $x(t), y(t)$.
- **Part 2**. Report a fundamental matrix $\Phi(t)$.
- **Part 3**. Use **Part 2** to find the exponential matrix e^{At}.

Chapter 6, Linear and Nonlinear Dynamical Systems

Problem (3a) [20%] Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node. Sub-classification into improper or proper node is not required.

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} * & * \\ & * \end{pmatrix} \vec{u}$$

Problem (3b) [30%] Consider the nonlinear dynamical system

$$\begin{aligned} x' &= *, \\ y' &= *. \end{aligned}$$

An equilibrium point is $x = *$, $y = *$. Compute the Jacobian matrix of the linearized system at this equilibrium point.

Problem (3c) [30%] Consider the nonlinear system $\begin{cases} x' &= *, \\ y' &= *. \end{cases}$

> **(Part 1)** Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\frac{d}{dt}\vec{u} = A\vec{u}$, where A is the Jacobian matrix of this system at $x = *$, $y = *$.
>
> **(Part 2)** Apply the Pasting Theorem to classify $x = 2$, $y = 0$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

Problem (3d) [20%] State the hypotheses and the conclusions of the *Pasting Theorem* used in problem **(3c)** above. Accuracy and completeness expected.