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Differential Equations 2280

Midterm Exam 2

Exam Date: 30 March 2018 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Chapter 3)** 96

A− −4 **(a)** [60%] Find by any applicable method the steady-state periodic solution for the mechanical system

$$x''(t) + 8\,x'(t) + 16\,x(t) = 289\,(\cos(t) - 10\sin(t))\,.$$

A **(b)** [40%] Variation of parameters can solve the electrical/mechanical problem

$$y''(x) + 8y'(x) + 16y(x) = f(x)$$

because of superposition $y = y_p + y_h$. Solve for the homogeneous solution $y_h(x)$ and then write the variation of parameters formula for a particular solution $y_p(x)$.

Expected in (b): (1) Characteristic equation and Euler atoms. (2) Wronskian, (3) Explicit formula for y_h. (4) Integral formula for y_p, simplified as much as possible, integrals unevaluated.

7810, $2890 - 289 = 2501$

(a)

$x = A\cos(t) + B\sin(t)$ $\qquad 16A = -2501$

$x' = -A\sin(t) + B\cos(t)$

$x'' = -A\cos(t) - B\sin(t)$

$-A\cos(t) - B\sin(t) - 8A\sin(t) + 8B\cos(t) + 16A\cos(t) + 16B\sin(t)$ $\qquad 2890 + 289 = 3179$

$-A + 8B + 16A = 289 \qquad 8B + 15A = 289$ ✓

Sign error $\quad -B + 8A + 16B = -2890 \qquad +8A + 15B = -2890$ $\qquad B = \frac{3179}{30}$

$30B = 3179$

$$\boxed{x = \frac{-2501}{16}\cos(t) + \frac{3179}{30}\sin(t)}$$

← wrong A, B : A = 95, B = −142

A = 95, B = −142

(b) $r^2 + 8r + 16$, atoms: xe^{-4x}, e^{-4x} ← $y_1 = xe^{-4x}$; $y_2 = e^{-4x}$

$$W(x) = \begin{vmatrix} xe^{-4x} & e^{-4x} \\ e^{-4x} - 4xe^{-4x} & -4e^{-4x} \end{vmatrix} = -e^{-8x}$$

$$y(x) = e^{-4x}\int \frac{-xe^{-4x}\,dx}{-e^{-8x}} + xe^{-4x}\int \frac{e^{-4x}\,dx}{-e^{-8x}} = \boxed{e^{-4x}\int xe^{4x}\,dx - xe^{-4x}\int e^{4x}\,dx}$$

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2. **(Laplace Theory)**

A− −3 **(a)** [50%] Assume $f(t)$ is of exponential order. Find $f(t)$ in the relation

$$\left(\frac{d}{ds}\mathcal{L}(f(t))\right)\Big|_{s\to(s-3)} + \frac{\mathcal{L}(t)}{s^2+4} + \mathcal{L}(\sin(t)) = 0.$$

A **(b)** [50%] Solve by Laplace's method

$$x''(t) + 4x'(t) + 3x(t) = 4\,e^{-t}, \quad x(0) = x'(0) = 0.$$

(a) $\left(\frac{d}{ds}\mathcal{L}(f(t))\right) = \mathcal{L}(-tf(t))$

$\frac{\mathcal{L}(t)}{s^2+4} = \frac{1}{s^2(s^2+4)}$ * $= \frac{a}{s^2} + \frac{b}{s^2+4} = \frac{1/4}{s^2} + \frac{-1/4}{s^2+4} = \mathcal{L}\left(t/4 - \frac{1}{8}\sin 2t\right)$

$\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$

↑ a function of s^2, using Heaviside

$+tf(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)} + \frac{1}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4s} - \frac{1}{4(s^2+4)} + \frac{1}{s^2+1}\right\}$

e^{3t}

$$\boxed{f(t) = \frac{1}{4t} - \frac{1}{8t}\sin 2t + \frac{1}{t}\sin t}$$

lost shift on LHS

(b) $s^2X + 4sX + 3X = \frac{4}{s+1} \Rightarrow X(s^2+4s+3) = \frac{4}{s+1}$ ✓

$X = \frac{4}{(s+1)^2(s+3)} = \frac{As+B}{(s+1)^2} + \frac{C}{s+3}$

$As^2 + Bs + 3As + 3B + Cs^2 + 2Cs + C$

$A + C = 0 \Rightarrow A = -C$

$3A + B + 2C = 0 \Rightarrow A = -B$

$3B + C = 4$

$B = C$

$4B = 4$

$B = 1 = C$

$A = -1$

$X = \frac{1-s}{(s+1)^2} + \frac{1}{s+3}$

$X = \frac{2}{(s+1)^2} + \frac{-1}{s+1} + \frac{1}{s+3}$

$$\boxed{x(t) = 2te^{-t} - e^{-t} + e^{-3t}}$$

3. (Laplace Theory) 100

(a) [40%] Solve $x''' + x'' = 0$, $x(0) = 1$, $x'(0) = 1$, $x''(0) = 0$ by Laplace's Method.

A

Expected in (a): (1) Laplace's method steps to find $\mathcal{L}(x(t))$, (2) Backward table steps to find $x(t)$, (3) An answer check.

(b) [60%] Solve the system $x' = 2x + y + 10$, $y' = x + 2y + 20$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

A

Expected in (b): (1) Laplace's method steps to find a system of linear algebraic equations for $\mathcal{L}(x(t))$ and $\mathcal{L}(y(t))$. (2) Cramer's Rule steps to find $\mathcal{L}(x(t))$ and $\mathcal{L}(y(t))$. (3) Backward Laplace table steps to find $x(t), y(t)$. (4) Check the answer for sanity, e.g., compute the Euler atoms.

(a) $x''' + x'' = 0 \quad x(0) = 1,\ x'(0) = 1,\ x''(0) = 0$

$s^3 X - s^2 - s + s^2 X - s - 1 = 0$

$(s^3 + s^2)X = s^2 + 2s + 1 \qquad X = \dfrac{(s+1)^{\cancel{2}}}{\cancel{(s+1)}s^2} = \dfrac{s+1}{s^2} = \dfrac{1}{s} + \dfrac{1}{s^2}$

$x(t) = 1 + t$

$x''(t) = 0 \qquad x''(t) + x'''(t) = 0$ ✓

$x'''(t) = 0$

$x(0) = 1$ ✓

$x'(0) = 1$ ✓

b) $sX = 2X + Y + \dfrac{10}{s} \qquad (s-2)X - Y = \dfrac{10}{s}$

$sY = X + 2Y + \dfrac{20}{s} \qquad (s-2)Y - X = \dfrac{20}{s}$

$(s-2)^2 X - (s-2)Y = \dfrac{10(s-2)}{s}$

$-X \quad (s-2)Y = \dfrac{20}{s}$

$X = \dfrac{10(s-2) + 20}{s(s^2 - 4s + 3)} = \dfrac{10}{(s-3)(s+1)}$

$x(t) = \mathcal{L}^{-1}\left(\dfrac{5}{s-3}\right) + \mathcal{L}^{-1}\left(\dfrac{-5}{s-1}\right)$

$(s-2)^2 X - X = \dfrac{10(s-2)}{s} + \dfrac{20}{s}$

$x(t) = 5e^{3t} + 5e^{t}$

~~$s^2 - 4s + 5$~~

$15e^{3t} - 5e^{t} = 10e^{3t} - 10e^{t} + y + 10$

X ~~$s^2 - 4s + 3 =$~~

$5e^{3t} + 5e^{t} - 10 = y$

$$\boxed{\begin{aligned} X &= 5e^{3t} - 5e^{t} \\ y &= 5e^{3t} + 5e^{t} - 10 \end{aligned}}$$

4. (Systems of Differential Equations)

The Eigenanalysis Method (section 5.2) says that, for a 3×3 system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$, the general solution is $\vec{u}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + c_3\vec{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$, $(\lambda_3, \vec{v}_3)$ are eigenpairs of A. Assume given the two 3×3 matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

A **(a)** [50%] Matrix A has only two eigenpairs. Display eigenanalysis details for A. Then explain why the Eigenanalysis Method fails.

A **(b)** [50%] Find by any method the general solution of $\frac{d}{dt}\vec{u}(t) = B\vec{u}(t)$.

$$\det(A-\lambda I) = \left|\begin{pmatrix} 2-\lambda & 1 & 2 \\ 1 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix}\right| = (3-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(4-4\lambda+\lambda^2-1)$$

$\cancel{4 \pm \sqrt{16-12}}$

$$= (3-\lambda)(\lambda^2-4\lambda+3) = (3-\lambda)(\lambda-1)(\lambda-3)$$

$$(A-3I) = \left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x - y + 2z = 0$

$x = y - 2z$

$$(A-I) = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Eigenanalysis fails because A has ~~only two eigenvalues~~ a row of zeros with one value that ~~only two eig~~ ~~vectors~~ another eigenvalue

(b) $\cancel{\det(B-\lambda I)} = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$ upper tria-gular, so eigenvalues are 2,2,3

Using method (4) in problem 5. Find u_2, u_3 by section 1.5 'shortcut'

$u_3' = 3u_3 \Rightarrow u_3 = c_3 e^{3t}$ ✓

$$(B-2I|0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \cancel{v_1 =} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$u_2' = 2u_2 \Rightarrow u_2 = c_2 e^{2t}$ ✓

$$\cancel{(B-3I|0) = \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)} \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$u_1' = 2u_1 + c_2 e^{2t}$

$u_1' - 2u_1 = c_2 e^{2t}$

integrating factor method

$(u_1 e^{-2t})' = c_2$ ✓

$u_1 = c_2 t e^{2t} + c_1 e^{2t}$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$ ✓

Solution to $\vec{u}' = B\vec{u}$

5. (Systems of Differential Equations)
A system $\frac{d}{dt}\vec{u} = A\vec{u}$ with A an $n \times n$ real matrix can be solved by many methods. Consider below **only these possibilities:**

(1) Cayley-Hamilton-Ziebur method, from section 4.2. (2) Eigenanalysis method from section 5.2. (3) Laplace's method, from chapter 7. (4) First order linear integrating factor method from section 1.5.

A

(a) [20%] A certain system $\frac{d}{dt}\vec{u} = A\vec{u}$ is supplied with 4×4 matrix A having four eigenpairs, but no real eigenvalues. It can be solved using the eigenanalysis method from section 5.2. Which of methods (1), (3), (4) might be possible to use? Don't do any details or illustrations: kindly write a sentence and explain your logic.

A

(b) [40%] Solve for the general solution by the Cayley-Hamilton-Ziebur shortcut (chapters 4 and 5):

$$\frac{d}{dt}\vec{u} = B\vec{u}, \qquad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad \textbf{(eigenvalues} = 2, 4).$$

A

(c) [40%] Choose any method applicable and solve for the general solution of system

$$\begin{cases} x'(t) &= x(t), \\ y'(t) &= y(t) + z(t), \\ z'(t) &= x(t). \end{cases}$$

(a) Laplaces method would work because it would just be a matter of solving algebraic equations instead of differential ones. CHZ would also work because it's just a faster method* of eigenanalysis. But (4) has limitations and is not strong enough.

(b) $(B - 2I) = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$ $\quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

* CHZ uses Euler atoms (real) and real vectors $\vec{d}_1, \ldots, \vec{d}_n$ avoiding burst of complex eigenvectors and Eulers exponent. formula for $e^{i\theta}$.

$(B - 4I) = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$ $\quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

CHZ: $\vec{u} = \vec{d}_1 e^{2t} + \vec{d}_2 e^{4t}$

$$\boxed{\vec{u} = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

← It is a fact That CHZ $\vec{d}_1, \vec{d}_2$ are eigenvectors in this case, of 2 real eigenvalues.

(c) $x'(t) = x(t) \Rightarrow x(t) = c_1 e^t$

$z'(t) = c_1 e^t \Rightarrow z(t) = c_1 e^t$

$y'(t) - y(t) = c_1 e^t \quad (y(t)e^{-t})' = c_1 \Rightarrow y(t) = c_1 t e^t + c_2 e^t$

using method (4)