Differential Equations 2280

Midterm Exam 2 with Solutions

Exam Date: 31 March 2017 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Chapter 3)**

(a) [70%] Find by any applicable method the steady-state periodic solution for the current equation $I'' + 2I' + 5I = 10\cos(t) - 100\sin(t)$.

(b) [30%] Linear algebra can find the solution of the current equation $I''+2I'+5I = 10\cos(t)-100\sin(t)$ having initial conditions $I(0) = 1$, $I'(0) = 0$. Write the linear algebraic equations for c_1, c_2, but to save time don't solve for c_1, c_2.

Answer:

Part (a) Answer: $I_{SS}(t) = \cos t - 2\sin t$.

Variation of Parameters.

Solve $x'' + 2x' + 5x = 0$ to get $x_h = c_1x_1 + c_2x_2$, $x_1 = e^{-t}\cos 2t$, $x_2 = e^{-t}\sin 2t$. Compute the Wronskian $W = x_1x_2' - x_1'x_2 = 4e^{-2t}$. Then for $f(t) = -10\sin(t)$,

$$x_p = x_1 \int x_2 \frac{-f}{W} dt + x_2 \int x_1 \frac{f}{W} dt.$$

The integrations are too difficult, so the method won't be pursued.

Undetermined Coefficients.

The trial solution by Rule I is $I = d_1\cos t + d_2\sin t$. The homogeneous solutions have exponential factors, therefore the Euler solution atoms in the trial solution cannot be solutions of the homogeneous problem, hence Rule II does not apply.

Substitute the trial solution into the non-homogeneous equation to obtain the answers $d_1 = 1$, $d_2 = -2$. The unique periodic solution I_{SS} is extracted from the general solution $I = I_h + I_p$ by crossing out all negative exponential terms (terms which limit to zero at infinity). Because $I_p = d_1\cos t + d_2\sin t = \cos t - 2\sin t$ and the homogeneous solution x_h has negative exponential terms, then

$$I_{SS} = \cos t - 2\sin t.$$

Laplace Theory.

Plan: Find the general solution, then extract the steady-state solution by dropping negative exponential terms. The computation can use initial data $I(0) = I'(0) = 0$, because every particular solution contains the terms of the steady-state solution. Some details:

$(s^2 + 2s + 5)\mathcal{L}(I) = \dfrac{-10}{s^2+1}$

$\mathcal{L}(I) = \dfrac{-10}{(s^2+1)(s^2+2s+5)}$

$\mathcal{L}(I) = \dfrac{-10}{(s^2+1)((s+1)^2+4)}$

$\mathcal{L}(I) = \dfrac{s-2}{s^2+1} - \dfrac{s}{(s+1)^2+4}$

$\mathcal{L}(I) = \dfrac{s}{s^2+1} - 2\dfrac{1}{s^2+1} - \dfrac{s+1}{(s+1)^2+4} + \dfrac{1}{2}\dfrac{2}{(s+1)^2+4}$

$\mathcal{L}(I) = \mathcal{L}(\cos t) - 2\mathcal{L}(\sin t) - \mathcal{L}(e^{-t}\cos 2t) + \dfrac{1}{2}\mathcal{L}(e^{-t}\sin 2t)$

$I(t) = \cos t - 2\sin t - e^{-t}\cos 2t + \dfrac{1}{2}e^{-t}\sin 2t$, by Lerch's Theorem.

Dropping the negative exponential terms gives the steady-state solution $I_{SS}(t) = \cos t - 2\sin t$.

Part (b) Answer: $y_p = \dfrac{x^2}{2} - \dfrac{x^3}{6}$.

Variation of Parameters.

Solve $y'' = 0$ to get $y_h = c_1y_1 + c_2y_2$, $y_1 = 1$, $y_2 = x$. Compute the Wronskian $W = y_1y_2' - y_1'y_2 = 1$. Then for $f(t) = 1 - x$,

$y_p = y_1 \int y_2 \frac{-f}{W} dx + y_2 \int y_1 \frac{f}{W} dx$,

$y_p = 1 \int -x(1-x)dx + x \int 1(1-x)dx$,

$y_p = -1(x^2/2 - x^3/3) + x(x - x^2/2)$,

$y_p = x^2/2 - x^3/6$.

This answer is checked by quadrature, applied twice to $y'' = 1 - x$ with initial conditions zero.

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2. **(Laplace Theory)**

(a) [40%] Assume $f(t)$ is of exponential order. Find $f(t)$ in the relation

$$\left.\frac{d^2}{ds^2}\mathcal{L}(f(t))\right|_{s\to(s-3)} = \frac{1}{s^2} + \mathcal{L}(t^2f(t) - t).$$

(b) [60%] Solve by Laplace's method $x'' + 2x' + x = e^{-t}$, $x(0) = x'(0) = 0$.

Answer:

(a)

$x(t) = -1/4\,\mathrm{e}^{-t} - 1/2\,\mathrm{e}^{-t}t + 1/4\,\mathrm{e}^{t}$

An intermediate step is $\mathcal{L}(x(t)) = \dfrac{1}{(s-1)(s+1)^2}$. The solution uses partial fractions $\dfrac{1}{(s-1)(s+1)^2} = \dfrac{A}{s-1} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$, with answers $A = 1/4$, $B = -1/4$, $C = -1/2$.

(b)

Replace by the shift theorem and the s-differentiation theorem the given equation by

$$\mathcal{L}\left((-t)f(t)e^{3t}\right) = \mathcal{L}(f(t) - t).$$

Then Lerch's theorem cancels $\mathcal{L}$ to give $-te^{3t}f(t) = f(t) - t$. Solve for

$$f(t) = \frac{t}{1 + te^{3t}}.$$

(c)

The main steps are:

$(s^2 + 4s + 4)\mathcal{L}(y(t)) = \mathcal{L}(f(t))$,

$\mathcal{L}(y(t)) = \dfrac{1}{(s+2)^2}\mathcal{L}(f(t))$,

$\mathcal{L}(y(t)) = \mathcal{L}(te^{-2t})\mathcal{L}(f(t))$, by the first shifting theorem,

$\mathcal{L}(y(t)) = \mathcal{L}(\text{convolution of } te^{-2t} \text{ and } f(t))$, by the Convolution Theorem,

$\mathcal{L}(y(t)) = \mathcal{L}\left(\int_0^t xe^{-2x}f(t-x)dx\right)$, insert definition of convolution,

$y(t) = \int_0^t xe^{-2x}f(t-x)dx$, by Lerch's Theorem.

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3. **(Laplace Theory)**

(a) [30%] Solve $\mathcal{L}(f(t)) = \dfrac{10/s}{(s^2+1)(s^2+5)}$ for $f(t)$.

(b) [30%] Solve $x''' + x' = 0$, $x(0) = 1$, $x'(0) = 1$, $x''(0) = 0$ by Laplace's Method.

(c) [40%] Solve the system $x' = 4x + y + 30$, $y' = x + 4y + 60$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

Answer:

(4a) Laplace's method explained.

The first step transforms the equation using the parts formula and initial data to get

$$(s+2)\mathcal{L}(x) = 1 + \mathcal{L}(e^t).$$

The forward Laplace table applies to evaluate $\mathcal{L}(e^t)$. Then write, after a division, the isolated formula for $\mathcal{L}(x)$:

$$\mathcal{L}(x) = \frac{1 + 1/(s-1)}{s+2} = \frac{s}{(s-1)(s+2)}.$$

Partial fraction methods plus the backward Laplace table imply

$$\mathcal{L}(x) = \frac{a}{s-1} + \frac{b}{s+2} = \mathcal{L}(ae^t + be^{-2t})$$

and then $x(t) = ae^t + be^{-2t}$ by Lerch's theorem. The constants are $a = 1/3$, $b = 2/3$.

(4b) $\mathcal{L}(f) = \frac{100}{(u+1)(u+4)} = \frac{100/3}{u+1} + \frac{-100/3}{u+4}$ where $u = s^2$. Then $\mathcal{L}(f) = \frac{100}{3}(\frac{1}{s^2+1} - \frac{1}{s^2+4}) = \frac{100}{3}\mathcal{L}(\sin t - \frac{1}{2}\sin 2t)$ implies $f(t) = \frac{100}{3}(\sin t - \frac{1}{2}\sin 2t)$.

(4c) $\mathcal{L}(f) = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+3} = \mathcal{L}(a + bt + ce^{-3t})$ implies $f(t) = a + bt + ce^{-3t}$. The constants, by Heaviside coverup, are $a = -1/9$, $b = 1/3$, $c = 1/9$.

(4d) $\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(e^{2t}\sin 3t)$ by the s-differentiation theorem. The first shifting theorem implies $\mathcal{L}(e^{2t}\sin 3t) = \mathcal{L}(\sin 3t)|_{s\to(s-2)}$. Finally, the forward table implies $\mathcal{L}(f) = \dfrac{d}{ds}\left(\dfrac{1}{(s-2)^2+9}\right) = \dfrac{-2(s-2)}{((s-2)^2+9)^2}$.

(4e) The answer is $x(t) = 1$, by guessing, then checking the answer. The Laplace details jump through hoops to arrive at $(s^3 + s^2)\mathcal{L}(x(t)) = s^2 + s$, or simply $\mathcal{L}(x(t)) = 1/s$. Then $x(t) = 1$.

(4f) The transformed system is

$$\begin{array}{rcrcl} (s-1)\mathcal{L}(x) & + & (-1)\mathcal{L}(y) & = & 0, \\ (-1)\mathcal{L}(x) & + & (s+1)\mathcal{L}(y) & = & \mathcal{L}(2). \end{array}$$

Then $\mathcal{L}(2) = 2/s$ and Cramer's Rule gives the formulas

$$\mathcal{L}(x) = \frac{2}{s(s^2-2)}, \quad \mathcal{L}(y) = \frac{2(s-1)}{s(s^2-2)}.$$

After partial fractions and the backward table,

$$x = -1 + \cosh(\sqrt{2}t), \quad y = \sqrt{2}\sinh(\sqrt{2}t) - \cosh(\sqrt{2}t) + 1.$$

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4. **(Systems of Differential Equations)**

The Eigenanalysis Method (section 5.2) says that, for a 3×3 system $\frac{d}{dt}\vec{u} = A\vec{u}$, the general solution is

$\vec{u}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $(\lambda_3, \mathbf{v}_3)$ are eigenpairs of A. Assume given the 3×3 matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

(a) [50%] Matrix A has only two eigenpairs. Display eigenanalysis details for A.
(b) [25%] It is impossible to apply the Eigenanalysis Method (stated above). Explain why.
(c) [25%] Display the solution of $\frac{d}{dt}\vec{u} = A\vec{u}$ in case A is 4×4 and has eigenvalues $2, -1, 3, 5$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Answer:
(a): The details should solve the equation $|A - \lambda I| = 0$ for three values $\lambda = 5, 4, 3$. Then solve the three systems $(A - \lambda I)\vec{v} = \vec{0}$ for eigenvector $\vec{v}$, for $\lambda = 5, 4, 3$.
The eigenpairs are

$$5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 4, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(b): The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(c): The eigenpairs are

$$6, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 7, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad 4, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

and The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{6t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

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5. **(Systems of Differential Equations)**
Systems $\frac{d}{dt}\vec{u} = A\vec{u}$ with A an $n \times n$ real matrix can be solved by the following methods:

(1) Cayley-Hamilton-Ziebur method, from section 4.2. (2) Eigenanalysis method from 5.2. (3) Laplace's method, from chapter 7. (4) Exponential matrix, from 5.6

(a) [50%] The eigenvalues are 3, 5 for the matrix $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$. Display the general solution of $\frac{d}{dt}\vec{u} = A\vec{u}$ according to the Cayley-Hamilton-Ziebur shortcut (textbook chapters 4,5).

(b) [10%] The 3×3 system $\frac{d}{dt}\vec{u} = A\vec{u}$ is supplied with matrix A having only two eigenpairs. It can be solved using the exponential matrix. What other methods are possible to use? Don't do any details, write a sentence.

(c) [10%] The 3×3 system $\frac{d}{dt}\vec{u} = A\vec{u}$ is supplied with matrix A having three eigenpairs, but only one real eigenvalue. It can be solved using the exponential matrix. What other methods are possible to use? Don't do any details, write a sentence.

(d) [30%] The 3×3 system $\frac{d}{dt}\vec{u} = A\vec{u}$ is given by $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Choose a method other than the exponential matrix and explain how you would solve for $\vec{u}$. It is not necessary to find the answer, but it is necessary to outline the method, not omitting any details.

Answer:
(a) **Cayley-Hamilton Ziebur Shortcut**. The method says that the components $x(t), y(t)$ of the solution to the system

$$\frac{d}{dt}\vec{u} = A\vec{u}, \quad \vec{u}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ are linear combinations of the Euler atoms found from the roots of the characteristic equation $|A - rI| = 0$. The roots are $r = 3, 5$ and the atoms are e^{3t}, e^{5t}. The scalar system is

$$\begin{cases} x'(t) &=& 4x(t) + y(t), \\ y'(t) &=& x(t) + 4y(t), \\ x(0) &=& 1, \\ y(0) &=& -1. \end{cases}$$

The C-H-Z method implies $x(t) = c_1 e^{3t} + c_2 e^{5t}$, but c_1, c_2 are not arbitrary constants: they are determined by the initial conditions $x(0) = 1, y(0) = -1$. Then $x' = 4x + y$ can be solved for y to obtain $y(t) = x'(t) - 4x(t)$. Substitute expression $x(t) = c_1 e^{3t} + c_2 e^{5t}$ into $y(t) = x'(t) - 4x(t)$ to obtain

$$y(t) = x'(t) - 4x(t) = 3c_1 e^{3t} + 5c_2 e^{5t} - 4(c_1 e^{3t} + c_2 e^{5t}) = -c_1 e^{3t} + c_2 e^{5t}.$$

Then

(1) $$\begin{cases} x(t) &=& c_1 e^{3t} &+& c_2 e^{5t}, \\ y(t) &=& -c_1 e^{3t} &+& c_2 e^{5t}. \end{cases}$$

Initial data $x(0) = 1, y(0) = -1$ are used in the last step, to evaluate c_1, c_2. Inserting these conditions produces a 2×2 linear system for c_1, c_2

$$\begin{cases} 0 &=& c_1 e^0 &+& c_2 e^0, \\ 0 &=& -c_1 e^0 &+& c_2 e^0. \end{cases}$$

The solution is $c_1 = 1$ and $c_2 = 0$, which implies the final answer $x(t) = e^{3t}$, $y(t) = -e^{3t}$.

Remark on Fundamental Matrices. The answer prior to evaluation of c_1, c_2 can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{5t} \\ -e^{3t} & e^{5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The matrix $\Phi(t) = \begin{pmatrix} e^{3t} & e^{5t} \\ -e^{3t} & e^{5t} \end{pmatrix}$ is called a **fundamental matrix**, because it is nonsingular and satisfies $\Phi' = A\Phi$ (its columns are solutions of $\frac{d}{dt}\vec{u} = A\vec{u}$). In terms of Φ,

$$e^{At} = \Phi(t)\Phi^{-1}(0).$$

This formula gives an alternative way to compute e^{At} by using the Cayley-Hamilton-Ziebur shortcut. Please observe that the columns of Φ are the formal partial derivatives of the vector solution $\vec{u}$ on the symbols c_1, c_2. Partial derivatives on symbols is a general method for discovering basis vectors. Therefore, Φ can be written directly from equations (**??**).