Name KEY

# Differential Equations 2280

**Midterm Exam 1 [8:35]**

**Wednesday, 25 February 2009**

**Instructions**: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**
   - **(a)** [25%] Solve $y' = \dfrac{3+x^2}{1+x^2}$.
   - **(b)** [25%] Solve $y' = (2\sin x + \cos x)(\sin x - 2\cos x)$.
   - **(c)** [25%] Solve $y' = \dfrac{x\tan\left(\ln(1+x^2)\right)}{1+x^2}$, $y(0) = 2$.
   - **(d)** [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(t^2 v(t)) = 0$, $v(2) = 10$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = -20$.

(a) $y = \int \frac{3+x^2}{1+x^2}\,dx = \int \frac{2\,dx}{1+x^2} + \int 1\,dx = 2\tan^{-1}(x) + x + C$

(b) $y = \int (2\sin x + \cos x)(2\sin x + \cos x)^{1}(-1)\,dx = \frac{-1}{2}(2\sin x + \cos x)^2 + C$

(c) $y = \int \frac{x\tan(\ln(1+x^2))}{1+x^2}\,dx$

$u = \ln(1+x^2)$

$du = \frac{2x}{1+x^2}\,dx$

$= \int \tan(u)\,\frac{du}{2}$

$= \frac{-1}{2}\ln|\cos(u)| + C$

$= -\frac{1}{2}\ln(\cos(\ln(1+x^2))) + C$

(d) $t^2 v(t) = C \Rightarrow 4v(2) = C \Rightarrow 40 = C$

$\boxed{v(t) = \frac{40}{t^2}}$

$x' = \frac{40}{t^2}$

$x = -40t^{-1} + C \Rightarrow -20 = -40/2 + C \Rightarrow C = 0$

$\boxed{x = -40/t}$

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**2. (Classification of Equations)**
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions $F$ and $G$.

**(a)** [40%] Check ([X]) the problems that can be put into separable form. No details expected.

| | |
|---|---|
| [X] $y' + xy = y(2x + e^x) + x^2 y$ | [X] $y' = (x-1)(y+1) + (1-x)y$ |
| [X] $y' = 2e^{2x-y}e^{3y} + 3e^{3x+2y}$ | [ ] $y' + x^2 e^y = xy$ |

**(b)** [10%] Is $y' + x(y+1) = ye^x + x$ separable? No details expected.

**(c)** [10%] Give an example of $y' = f(x,y)$ which is separable and linear but not quadrature. No details expected.

**(d)** [40%] Apply tests to show that $y' = x + e^y$ is not separable and not linear. Supply all details.

(a) $y' + xy = 2xy + e^x y + x^2 y$ Linear, Separable

$y' = 2e^{2x}e^{2y} + 3e^{3x}e^{2y}$ Separable

$y' = xy - y + x - 1 + y - xy = x - 1$ SLQ

$y' = -x^2 e^y + xy$ not S, Q or L

(b) $y' = ye^x + x - xy - x = ye^x - xy = y(e^x - x)$

yes, Separable.

(c) $y' = xy$

(d) $f(x,y) = x + e^y$

$\frac{f_x}{f} = \frac{1}{x + e^y}$ not indep of $y$ $\Rightarrow$ not Separable

$f_y = e^y$ not indep of $y$ $\Rightarrow$ not linear

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**3. (Solve a Separable Equation)**

Given $(x+3)(y+1)y' = \left((x+3)e^{-x+2} + 3x^2 + 2\right)(y-1)(y+2)$.

Find a non-equilibrium solution in implicit form.

To save time, **do not solve** for $y$ explicitly and **do not solve** for equilibrium solutions.

$$\frac{y+1}{(y-1)(y+2)}\,y' = e^{2-x} + \frac{3x^2+2}{x+3}$$

$$\left(\frac{A}{y-1} + \frac{B}{y+2}\right)y' = e^{2-x} + 3x - 9 + \frac{29}{x+3}$$

integrate

$$\frac{2}{3}\ln|y-1| + \frac{1}{3}\ln|y+2| = -e^{-x+2} + \frac{3}{2}x^2 - 9x + 29\ln|x+3| + C$$

Long Division

$$\begin{array}{r} 3x - 9 \\ x+3 \overline{)\,3x^2+2} \\ \underline{3x^2+9x} \\ -9x+2 \\ \underline{-9x-27} \\ 29 \end{array}$$

partial fractions

$y+1 = A(y+2) + B(y-1)$

$-1 = -3B$

$2 = 3A$

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**4. (Linear Equations)**

**(a)** [60%] Solve the linear model $5x'(t) = -160 + \dfrac{25}{2t+3}x(t)$, $x(0) = 32$. Show all integrating factor steps.

**(b)** [20%] Solve the homogeneous equation $\dfrac{dy}{dx} - (2x)y = 0$.

**(c)** [20%] Solve $5\dfrac{dy}{dx} + 10y = 7$ using the superposition principle $y = y_h + y_p$. Expected are answers for $y_h$ and $y_p$.

(a) $x' + \dfrac{-5}{2t+3} = \dfrac{-160}{5}$, $\quad x(0) = 32$

$u = \int \dfrac{-5}{2t+3}\,dt$

$u = -\dfrac{5}{2}\ln|2t+3|$

$e^u = (2t+3)^{-5/2}$

$(e^u x)' = -32e^u$

$e^u x = -32\int (2t+3)^{-5/2}\,dt$

$= -32\,\dfrac{(2t+3)^{-3/2}}{(-3/2)(2)} + C$

$x = \dfrac{32}{3}(2t+3) + C(2t+3)^{5/2} \longrightarrow 32 = \dfrac{32}{3}(0+3) + C\,3^{5/2}$

$\longrightarrow C = 0$

$\boxed{x = \dfrac{64}{3}t + 32}$

(b) $y = \dfrac{C}{e^{-x^2}}$

(c) $y = \dfrac{7}{10} + \dfrac{C}{e^{2x}}$

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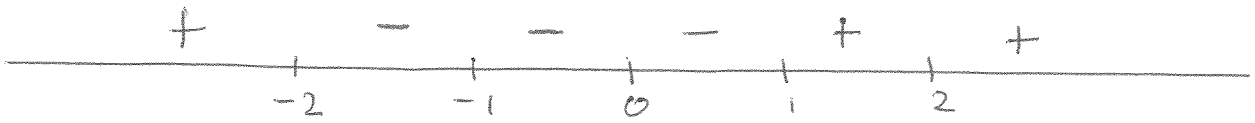
**5. (Stability)**

**(a)** [50%] Draw a phase line diagram for the differential equation

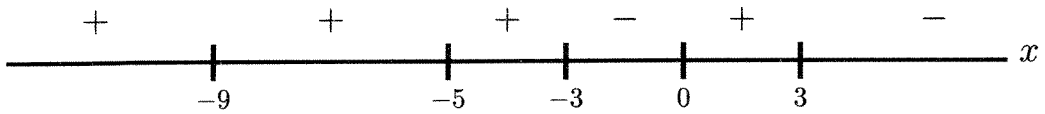
$$\frac{dx}{dt} = \left(\ln(1+5x^2)\right)^{1/5} (|2x-1|-3)^3 (2+x)^2 (4-x^2)(1-x^2)^3 e^{\cos x}.$$

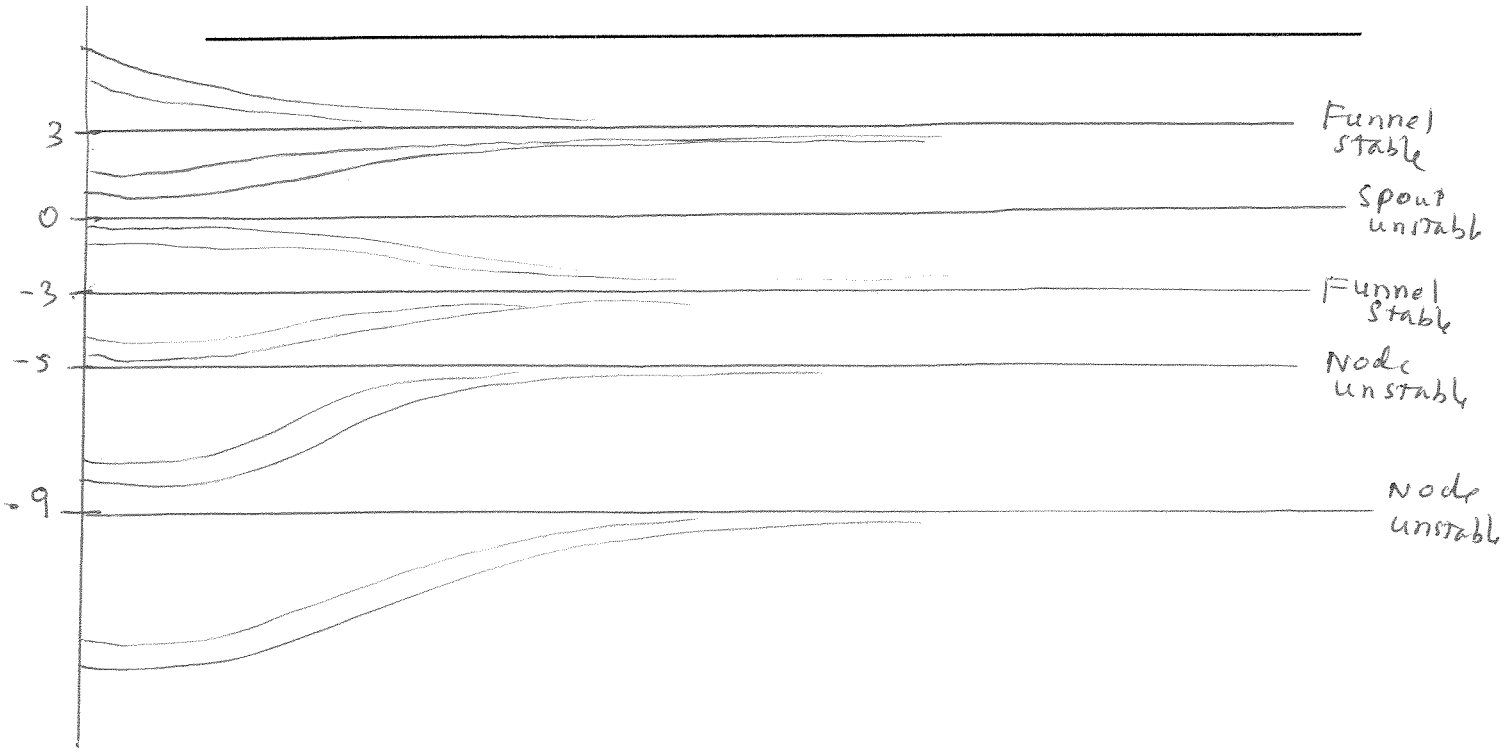
$x = 0$
$2x-1-3=0$
$2x-1+3=0$
$x+2=0$
$x-2=0$
$x=1$
$x=-1$

Expected in the phase line diagram are equilibrium points and signs of $dx/dt$.




**(b)** [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.

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