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Differential Equations 2280

Midterm Exam 1

Exam Date: Friday, 26 February 2016 at 12:50pm

Instructions: This in-class exam is designed for 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

(a) [40%] Solve $y' = \dfrac{2x^3}{1+x^2}$.

(b) [60%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(e^{-t}v(t)) = 2\,e^t$, $v(0) = 5$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = 2$.

Solution to Problem 1.

(a) Answer $y = x^2 - \ln\left(x^2+1\right) + c$. The integral of $F(x) = \frac{2x^3}{1+x^2}$ is found by substitution $u = 1+x^2$, resulting in the new integration problem $\int F\,dx = \int \frac{u-1}{u}du = \int(1)du - \int \frac{du}{u}$.

(b) Velocity $v(t) = 2e^{2t} + 3e^t$ by quadrature. Integrate $x'(t) = 2e^{2t} + 3e^t$ with $x(0) = 2$ to obtain position $x(t) = e^{2t} + 3e^t - 2$.

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2. (Classification of Equations)
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [40%] The equation $y' + x(y + 3) = ye^x + 3x$ is separable. Provide formulas for $F(x)$ and $G(y)$.

(b) [60%] Apply partial derivative tests to show that $y' = x + y$ is linear but not separable. Supply all details.

Solution to Problem 2.
(a) The equation is $y' = ye^x - xy = (e^x - x)y$. Then $F(x) = e^x - x$, $G(y) = y$.

(b) Let $f(x, y) = x+y$. Then $\partial f/\partial y = 1$, which is independent of y, hence the equation $y' = f(x, y)$ is linear. The negative test is $\frac{\partial f/\partial y}{f}$ depends on x. In this case, the fraction is

$$\frac{\partial f/\partial y}{f} = \frac{1}{f} = \frac{1}{x+y}.$$

At $y = 0$, this reduces to $1/x$, which depends on x, therefore the equation $y' = f(x, y)$ is not separable. Symmetrically, the test f_x/f depends on y implies $y' = f(x, y)$ is not separable.

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3. (Solve a Separable Equation)
Given $(5y + 10)y' = \left(xe^{-x} + \sin(x)\cos(x)\right)(y^2 + 3y - 4)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

Solution to Problem 3.
The solution by separation of variables identifies the separated equation $y' = F(x)G(y)$ using

$$F(x) = xe^{-x} + \sin(x)\cos(x), \quad G(y) = \frac{y^2 + 3y - 4}{5y + 10}.$$

The integral of F is done by parts and also by u-substitution.

$$\begin{aligned}
\textstyle\int F dx &= \textstyle\int xe^{-x}\,dx + \int \sin(x)\cos(x)\,dx \\
&= I_1 + I_2. \\
I_1 &= \textstyle\int xe^{-x}\,dx \\
&= -xe^{-x} - \textstyle\int e^{-x}\,dx, \quad \text{parts} \quad u = x, dv = e^{-x}dx, \\
&= -xe^{-x} - e^{-x} + c_1 \\
I_2 &= \textstyle\int \sin(x)\cos(x)\,dx \\
&= \textstyle\int u\,du, \quad u = \sin(x), du = \cos(x)dx, \\
&= u^2/2 + c_2 \\
&= \tfrac{1}{2}\sin^2(x) + c_2
\end{aligned}$$

Then $\displaystyle\int F dx = -xe^{-x} - e^{-x} + \frac{1}{2}\sin^2(x) + c_3.$

The integral of $1/G(y)$ requires partial fractions. The details:

$$\begin{aligned}
\int \frac{dx}{G(y(x))} &= \int \frac{5u + 10}{u^2 + 3u - 4}\,du, \quad u = y(x), du = y'(x)dx, \\
&= \int \frac{5u + 10}{(u + 4)(u - 1)}\,du \\
&= \int \frac{A}{u + 4} + \frac{B}{u - 1}\,du, \quad A, B \quad \text{determined later}, \\
&= A\ln|u + 4| + B\ln|u - 1| + c_4
\end{aligned}$$

The partial fraction problem

$$\frac{5u + 10}{(u + 4)(u - 1)} = \frac{A}{u + 4} + \frac{B}{u - 1}$$

can be solved in a variety of ways, with answer $A = \frac{-20+10}{-5} = 2$ and $B = \frac{15}{5} = 3$. The final implicit solution is obtained from $\int \frac{dx}{G(y(x))} = \int F(x)dx$, which gives the equation

$$2\ln|y + 4| + 3\ln|y - 1| = -xe^{-x} - e^{-x} + \frac{1}{2}\sin^2(x) + c.$$

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4. (Linear Equations)

(a) [60%] Solve the linear model $2x'(t) = -64 + \frac{10}{3t+2}x(t)$, $x(0) = 32$. Show all integrating factor steps.
(b) [20%] Solve $\frac{dy}{dx} - (\cos(x))y = 0$ using the homogeneous linear equation shortcut.
(c) [20%] Solve $5\frac{dy}{dx} - 7y = 10$ using the superposition principle $y = y_h + y_p$ shortcut. Expected are answers for y_h and y_p.

Solution to Problem 4.
(a) The answer is $v(t) = 32 + 48t$. The details:

$$\begin{aligned}
&v'(t) = -32 + \frac{5}{3t+2}v(t),\\
&v'(t) + \frac{-5}{3t+2}v(t) = -32, \quad \text{standard form } v' + p(t)v = q(t)\\
&p(t) = \frac{-5}{3t+2},\\
&W = e^{\int p\,dt}, \quad \text{integrating factor}\\
&W = e^u, \quad u = \textstyle\int p\,dt = -\frac{5}{3}\ln|3t+2| = \ln\left(|3t+2|^{-5/3}\right)\\
&W = (3t+2)^{-5/3}, \quad \text{Final choice for } W.
\end{aligned}$$

Then replace the left side of $v' + pv = q$ by $(vW)'/W$ to obtain

$$\begin{aligned}
&v'(t) + \frac{-5}{3t+2}v(t) = -32, \quad \text{standard form } v' + p(t)v = q(t)\\
&\frac{(vW)'}{W} = -32, \quad \text{Replace left side by quotient } (vW)'/W\\
&(vW)' = -32W, \quad \text{cross-multiply}\\
&vW = -32\textstyle\int W\,dt, \quad \text{quadrature step.}
\end{aligned}$$

The evaluation of the integral is from the power rule:

$$\int -32W\,dt = -32\int (3t+2)^{-5/3}dt = -32\frac{(3t+2)^{-2/3}}{(-2/3)(3)} + c.$$

Division by $W = (3t+2)^{-5/3}$ then gives the general solution

$$v(t) = \frac{c}{W} - \frac{32}{-2}(3t+2)^{-2/3}(3t+2)^{5/3}.$$

Constant c evaluates to $c = 0$ because of initial condition $v(0) = 32$. Then

$$v(t) = \frac{32}{-2}(3t+2)^{-2/3}(3t+2)^{5/3} = 16(3t+2)^{-\frac{2}{3}+\frac{5}{3}} = 16(3t+2).$$

(b) The answer is y = constant divided by the integrating factor: $y = \frac{c}{W}$. Because $W = e^u$ where $u = \int -\cos(x)dx = -\sin x$, then $y = ce^{\sin x}$.

(c) The equilibrium solution (a constant solution) is $y_p = -\frac{10}{7}$. The homogeneous solution is $y_h = ce^{7x/5}$ = constant divided by the integrating factor. Then $y = y_p + y_h = -\frac{10}{7} + ce^{7x/5}$.

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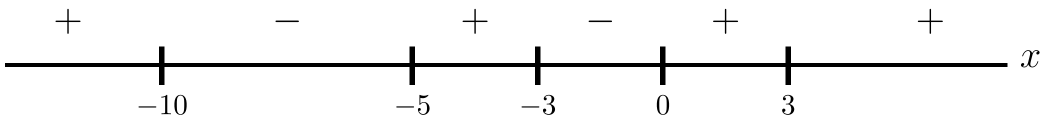
5. (Stability)

Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase portrait with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.



Solution to Problem 5.
The graphic is drawn using increasing and decreasing curves, which may or may not be depicted with turning points. The rules:

1. A curve drawn between equilibria is increasing if the sign is PLUS.
2. A curve drawn between equilibria is decreasing if the sign is MINUS.
3. Label: FUNNEL, STABLE
   The signs left to right are PLUS MINUS crossing the equilibrium point.
4. Label: SPOUT, UNSTABLE
   The signs left to right are MINUS PLUS crossing the equilibrium point.
5. Label: NODE, UNSTABLE
   The signs left to right are PLUS PLUS crossing the equilibrium point, or
   The signs left to right are MINUS MINUS crossing the equilibrium point.

The answer:

$x = -10$: FUNNEL, STABLE
$x = -5$: SPOUT, UNSTABLE
$x = -3$: FUNNEL, STABLE
$x = 0$: SPOUT, UNSTABLE
$x = 3$: NODE, UNSTABLE

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6. (ch3)

Using Euler's theorem on Euler solution atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c).

(a) [40%] Find a constant coefficient differential equation $ay'' + by' + cy = 0$ which has particular solutions $-5e^{-x} + xe^{-x}$, $10e^{-x} + xe^{-x}$.

(b) [30%] Given characteristic equation $r(r - 2)(r^3 + 4r)(r^2 + 2r + 37) = 0$, solve the differential equation.

(c) [30%] Given $mx''(t) + cx'(t) + kx(t) = 0$, which represents an unforced damped spring-mass system. Assume $m = 4$, $c = 4$, $k = 129$. Classify the equation as over-damped, critically damped or under-damped. Illustrate in a spring-mass-dashpot drawing the assignment of physical constants m, c, k and the initial conditions $x(0) = 1$, $x'(0) = 0$.

Solution to Problem 6.

6(a)

Multiply the first solution by 2 and add it to the second solution. Then Euler atom xe^{-x} is a solution, which implies that $r = -1$ is a double root of the characteristic equation. Then the characteristic equation should be $(r - (-1))(r - (-1)) = 0$, or $r^2 + 2r + 1 = 0$. The differential equation is $y'' + 2y' + y = 0$.

6(b)

The characteristic equation factors into $r^2(r - 2)(r^2 + 4)((r + 1)^2 + 36) = 0$ with roots $r = 0, 0; 2; \pm 2i; -1 \pm 6i$. Then y is a linear combination of the Euler solution atoms:

$1, x, e^{2x}, \cos(2x), \sin(2x); e^{-x}\cos(6x), e^{-x}\sin(6x)$.

6(c)

Use $4r^2 + 4r + 129 = 0$ and the quadratic formula to obtain roots $r = -1/2 + 4\sqrt{2}i, -1/2 - 4\sqrt{2}i$ and Euler solution atoms $e^{-x/2}\cos 4\sqrt{2}t, e^{-x/2}\sin 4\sqrt{2}t$. Then y is a linear combination of these two solution atoms, and it oscillates, therefore the classification is **under-damped**. The illustration shows a spring, a dashpot and a mass with labels k, c, m. Initial conditions mean mass elongation $x = 1$, at rest.

A **dashpot** is represented as a cylinder and piston with rod, the rod attached to the mass. Variable x is positive in the down direction and negative in the up direction. The equilibrium position is $x = 0$.

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7. (ch3)

Determine for $y^{(3)}+y^{(2)} = x+2e^{-x}+\sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate the undetermined coefficients**! The trial solution should be the one with fewest Euler solution atoms.

Solution to Problem 7.

The homogeneous equation $y^{(3)} + y^{(2)} = 0$ has solution $y_h = c_1 + c_2x + c_3e^{-x}$, because the characteristic polynomial has roots 0, 0, −1.

1 Rule I constructs an initial trial solution y from the list of independent Euler solution atoms

$$e^{-x}, \quad 1, \quad x, \quad \cos x, \quad \sin x.$$

Linear combinations of these atoms are supposed to reproduce, by assignment of constants, all derivatives of $f(x) = x + 2e^{-x} + \sin x$, which is the right side of the differential equation. Each of y_1 to y_4 in the display below is constructed to have the same **base atom**, which is the Euler atom obtained by stripping the power of x. For example, Euler solution atom xe^{0x} (or x, because $e^{0x} = 1$) strips to base atom e^{0x} or 1.

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1e^{-x}, \\ y_2 &= d_2 + d_3x, \\ y_3 &= d_4 \cos x, \\ y_4 &= d_5 \sin x. \end{aligned}$$

2 Rule II is applied individually to each of y_1, y_2, y_3, y_4 to give the **corrected trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1xe^{-x}, \\ y_2 &= x^2(d_2 + d_3x), \\ y_3 &= d_4 \cos x, \\ y_4 &= d_5 \sin x. \end{aligned}$$

The powers of x multiplied in each case are selected to eliminate terms in the initial trial solution which duplicate homogeneous equation Euler solution atoms. For instance, $y_1 = d_1e^{-x}$ is **in conflict** with the homogeneous equation, because e^{-x} is a common Euler atom of both y_1 and the homogeneous solution ($y_h = c_1 + c_2x + c_3e^{-x}$). Then Rule II multiplies y_1 by x to obtain the replacement $y_1 = d_1xe^{-x}$. This new term is again subjected to the Rule II test: Does y_1 contain an Euler atom of the homogeneous equation? The answer is NO, so the x-multiplication stops and the term y_1 is finished. We go on to the remaining terms, in the same way. Term y_2 needs two x-multiplications. The factor used after so many x-multiplications is exactly x^s of the Edwards-Penney table, where s is the multiplicity of the characteristic equation root r that produced the conflicting atom in the homogeneous solution y_h. The atoms in terms y_3, y_4 are not solutions of the homogeneous equation, therefore y_3, y_4 are unaltered by Rule II.