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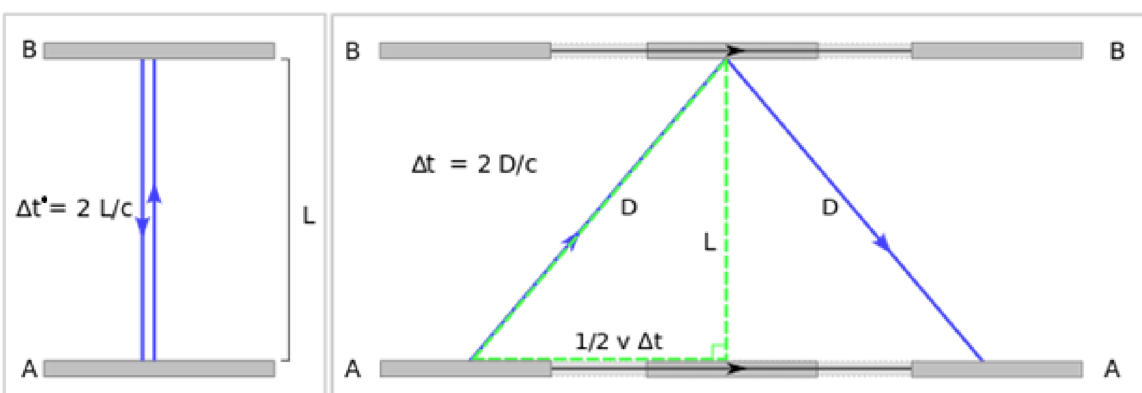
Semester project

Math 2270-002

Linear Algebra in Special Relativity

In 1905, Einstein presented his theory on special relativity that challenged classical Newtonian and Galilean physics. Einstein claimed a "God's eye view" of the universe was impossible and views could only be compared relative to another's view. The theory of special relativity begins with two postulates. First, that the laws of physics are the same for all uniformly moving (unaccelerated) observers. Second, that the speed of light in a vacuum is a constant for all observers regardless of their motion relative to the source. These two postulates came with very significant implications.

First, time is relative to your frame. The phenomena is more commonly known by the name of time dilation. The conversion is derived through a simple thought experiment with the set up as follows. A is a cart with B being a mirror. Time is measured for how long it takes for light (the blue line) takes to travel from the cart to the mirror and back to the cart from two different reference frames. The first frame is scene from the frame moving along with the cart and the second frame on the right is scene relative to someone watching the cart zip by.



In the first frame, the time it takes for the light to hit the mirror and bounce back to the cart is a simple calculation of distance divided by time. So $\Delta t' = 2L/c$, where c is equal to the speed of light through a vacuum. The second frame sets up two right triangles. Using just the left triangle we can form the equation $L^2 + \left(\frac{v\Delta t}{2}\right)^2 = \left(\frac{c\Delta t}{2}\right)^2$ by use of Pythagorean's theorem. And this simplifies to $\Delta t = \gamma \Delta t'$ where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

The second implication that applies is length contraction. Length contraction can be derived through a similar thought experiment, just rotated 90 degrees. The relation is, maintaining the convention of primed variables representing frames that are perceived to be in motion, $L' = \gamma L$. Using the above definition of γ and L represents length. From the length contraction and time dilation equations a set of transformation were formed. Again the primed frame is moving with velocity v relative to the unprimed frame.

$x' = \gamma(x - \beta ct)$ $\qquad$ $ct' = \gamma(ct - \beta x)$

$x = \gamma(x' + \beta ct')$ $\qquad$ $ct = \gamma(ct' + \beta x')$

Converting each individual coordinate and time for every event in one frame to another frame can become cumbersome. Fortunately, linear transformations apply and shorten this process. Just as in linear algebra where points are transformed from one space to another by use of a matrix the same can be done when transforming coordinates from one frame of reference to another. For simplicity the conversion will be done assuming all motion is along the x-axis.

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}$$

Example:

One sprinter is running with the velocity of 4/5c towards a barn. He is holding a stick horizontally as he runs. The stick has a length of 5 meters. The barn is 3 meters long. First find the length of the stick in the reference frame of someone standing at the entrance of the barn.

Stick length = 5m in reference frame of runner = L'

From the length contraction explained above we have L' = γL so L'/γ = L

L = $\frac{5m}{\frac{1}{\sqrt{1-\left(\frac{4}{5}\right)^2}}}$ = 3 meters

Now you can see that clearly for the by stander that the pole will enter into the barn. Say at time equals 0 the stick is in the barn and the doors close and open instantaneously to prove it. Next, find the length of the barn relative to the runner with the 5 meter stick.

γ = 5/3   γL = L'   L' = 3m

L = (3m)/(5/3) = 1.8 meters long

Clearly, a 5 meter stick will not fit into a 1.8 meter barn. So, what does the runner see if, as stated above, the doors close instantaneously when the stick enters? Does the door break the stick? Take the space time coordinates of the when the front end of the pole reaches the far door of the barn and when the back end of the pole enters the barn. We will call these event A and B respectively. We will say both frames coincide at time equals 0 and x will equal zero at the back door of the barn for simplicity. So event A in the reference frame of the observer at the barn door occurs at (0,0,0,0), where the coordinates are in the order (ct,x,y,z). That means that event B occurs at (0,-3,0,0). So, we will use the matrix transformation as explained above to do the math. Maple code is attached.

In the attached code we find that event A in the reference frame of the runner occurred at (0,0,0,0) and that event B in the reference frame of the runner occurred at (4,-5,0,0). So, in other words the doors are not seen to close at the same time in the reference frame of the runner. The first door instantaneously closes and opens just before the front end of the stick leaves the barn and then as time passes, until ct' = 4, the back end of the stick enters the barn just before the door instantaneously closes and opens.

Linear transformations allow for quick interchange of information of space-time coordinates and other information in special relativity. There is also a linear transformation for other quantities such as energy and momentum. These transformations allow for quick work and easier interpretation of what is occurring in the problem.