Applications of Linear Algebra in Political Science: Candidate Preferences in Elections

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In democratic states with liberal elections, voters almost always have multiple candidates to choose from. These candidates typically represent different political parties, ideologies, or voting pacts. Voters have transitive preferences for these multiple candidates. Due to a variety of factors, an individual's most preferred candidate may not be elected. Electoral rules are often responsible for diverging voting preferences from actual votes cast. Individuals often know that their vote actually counting is contingent on how others vote and electoral rules. As a result, individuals anticipate such electoral rules and potential for certain candidates and adjust their actual vote (which may depart from the individual's true preference). Linear algebra has been applied to the problem of voting preferences in the past to determine ways that a voter (or group of voters) can change their behavior to elect certain preferred candidates.

Linear Algebra in Election Preferences

This project will investigate and report on the mathematics used to analyze the preferences of voters for candidates in elections. The voting layout is put into a majority cycle, however, since these cycles are complicated to read and understand. They can be deconstructed into subsets that represent an individuals' preferences. These cycles are then added to linear combinations, which is where linear algebra comes into the equation. The majority cycle is decomposed into a cyclic and acyclic vectors. Through manipulation of the vectors, we can find the basis of the linear combination. This project will discuss, compute and analyze each of these aspects of linear algebra, and how they are applied to the voting system. The reason mathematicians found interest in creating these voting paradoxes is due to the easy manipulation of results based on the desired outcome of the person conducting the vote.

Example Linear Algebra in Election Preferences

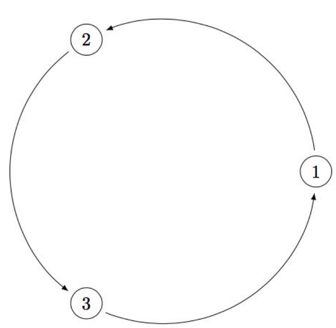
In American elections, people have two primary choices and usually a third independent choice. People tend to have intransitive ordered preferences. While we can say that we know how many people have what preferences, many of these preferences will cancel out. Thus, we want to know how to model the toal preferences while doing the least amount of math. Ideally, we would have a set of a set of basis vectors that we can multiply by each number of preference to get the total system of preferences.

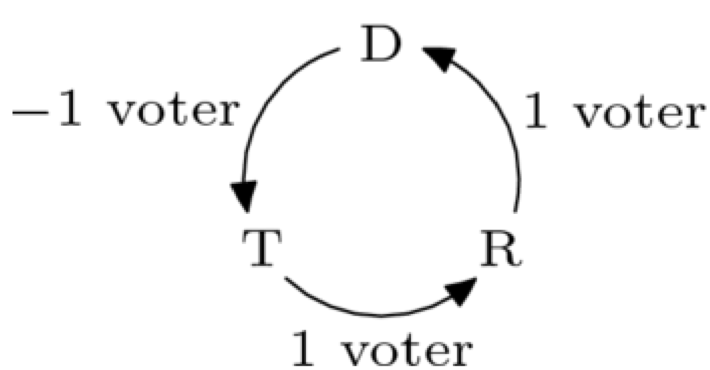
Preference order	*Number with that preference*
Democrat > Republican > Third	3
Democrat > Third > Republican	3
Republican > Democrat > Third	5
Republican > Third > Democrat	7
Third > Democrat > Republican	2
Third > Republican > Democrat	1

Total: 21

The image below represents the cyclical voting preferences of each individual. The total system then can be described by adding the total number of individuals who prefer candidate 1 (Democrat) to 2 (Republican), 2 to 3 (third), and 3 to 1. This may seem like a simple exercise, but with many more votes and voting preferences, this can get very complicated.

-5 people prefer Democrat to Republican (1 to 2)
-9 people prefer Republican to Third (2 to 3)
-1 people prefer Third to Democrat (3 to 1)





These preferences can then be broken down into individual preferences (right image, sourced from http://joshua.smcvt.edu/linearalgebra/book.pdf). The intuition behind this is that this voter actually prefers the third candidate to the Democratic candidate (hence the negative sign). This model can be used to represent all voters individual preferences, but thus arises the Condorcet

Paradox. How do individual non-cyclical preferences add up to a cyclical preference? To get the total model then, we require both a cyclical and acyclical model of individual preferences. There are only three possible models of an individual's cyclical preferences going counterclockwise.

(1)
$< -1, 1, 1 >, \ < -1, -1, 1 >, \ < 1, -1, -1 >$

Next, we will try to decompose a single vote vector <-1,1,1> into two parts: cyclic and acyclic systems. This can then be done for all three of the possible cyclical preference models. A vector in R^3 is purely cyclic if Vector C $= \ < a, a, a >$ given that a is part of R (all real numbers).

To find the acyclical system, we need to find C perp, where:
(2)
C perp = $< C_1, C_2, C_3 > \ * \ < a, a, a > \ = \ 0$

The basis can be found by first finding:
(3)
$C_2 * \ < -1, 1, 0 > \ + C_3 * \ < -1, 0, 1 >$ and $a * < 1, 1, 1 > \ = \ C$

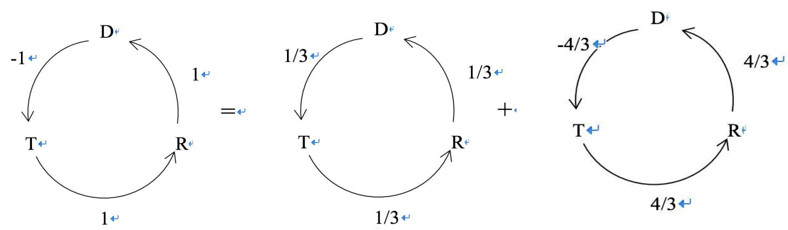
Thus, the basis is:
(4)
$< -1, 1, 0 >, \ < -1, 0, 1 >, \ and < 1, 1, 1 >.$

The vector solution with Strang's special solution to an individual voter are:
(5)
$C_1 - C_2 - C_3 = -1$
$C_1 + C_2 = 1$
$C_1 + C_3 = 1$

(6)
C_1 = ⅓
C_2 = ⅔
C_3 = ⅔

This suggests that
(7)
$< -1, 1, 1 > \ = (1/3) * < 1, 1, 1 > \ + (2/3) * < -1, 1, 0 > \ + (2/3) * < -1, 0, 1 > \ = \ < 1/3, 1/3, /1, 3 > \ +$
$< -4/3, 4/3, 4/3 >$

Giving us both the cyclical and acyclical components. This can be replicated for the 3 types of voter preferences, and by flipping signs, would also allow us to find the negative spins of each voter preference. Finding the overall preference is then just a matter of adding up these cyclical and acyclical preferences. By cancelling out both clockwise and counterclockwise “spins” if there are remaining preferences in just the clockwise or counterclockwise spin, we will have a paradox. Otherwise, there is no paradox.



Conclusion

Linear Algebra helps us understand total electoral preferences and potential paradoxes. Specifically, intransitive (or acyclical) preferences can sometimes add up to transitive (or cyclical) preferences. To determine if this is the case, it is helpful to have both a cyclical and acyclical representation of individual voter preferences. We used linear algebra to find both of these components, and showed that they can be used to determine a potential paradox.