

MATH 2270-2 Final Exam Spring 2016

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QUESTION	VALUE	SCORE	
1	60	57	-3
2	40		A
3	30		A
4	30		
5	20		A
6	40		A
7	30		A
8	40		A
9	30		A
10	30		A
11	30		A
12	30		A
13	30		A
14	40		A or A- -1
15	20		D+ -8
16	20		A
17	20		A- or A -1
TOTAL	540	500	-10

98.0%

No books, notes or electronic devices, please.

The questions have credits which reflect the time required to write the solution.

If you must write a solution out of order or on the back side, then supply a road map.

Solutions are expected to include readable and convincing details. A correct answer without details earns 25%.

Expect about 3 to 10 minutes per problem. Final exam problems may have multiple parts.

1. (Chapter 1: 60 points) Consider the system $A\vec{u} = \vec{b}$ with $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ defined by

$$2x_1 + 3x_2 + 4x_3 + x_4 = 2$$

$$4x_1 + 3x_2 + 8x_3 + x_4 = 4$$

$$6x_1 + 3x_2 + 8x_3 + x_4 = 2$$

Solve the following parts:

A (a) [10%] Find the reduced row echelon form of the augmented matrix.

A (b) [10%] Identify the free variables and the lead variables.

-3 C+ (c) [10%] Display a vector formula for a particular solution \vec{u}_p .

would be correct (d) [10%] Display a vector formula for the homogeneous solution \vec{u}_h .

(e) [10%] Identify each of Strang's Special Solutions.

(f) [10%] Display the vector general solution \vec{u} , using superposition.

Augmented matrix:

$$\left(\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 2 \\ 4 & 3 & 8 & 1 & 4 \\ 6 & 3 & 8 & 1 & 2 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 1 \\ 0 & -3 & 0 & -1 & 0 \\ 0 & -6 & -4 & -2 & -4 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 3 & 2 & 1 & 2 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 2 & 0 & 2 \end{array} \right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array} \right) \rightarrow \text{reduced row echelon form}$$

lead variable: x_1, x_2, x_3 free variable x_4

So particular solution $\vec{u}_p = \begin{bmatrix} -1 \\ -\frac{1}{3} \\ 0 \\ 0 \end{bmatrix}$

homogeneous solution $\vec{u}_h = t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ as $t \in \mathbb{R}$

Strang's special solution $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

vector general solution $\begin{bmatrix} -1 \\ -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{3} \\ 0 \\ x_4 \text{ free} \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{3} \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

2. (Chapter 2: 40 points)

(a) [10%] Describe for $n \times n$ matrices two different methods for finding the matrix inverse.

(b) [20%] Apply the two methods to find the inverse of the matrix $A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}$.

(c) [10%] Find the inverse of the transpose of the matrix in part (b).

(a) (1) using $A^{-1} = \frac{\text{adj}A}{|A|}$

(2) using $(A|I) \sim (I|A^{-1})$

(b) (1) $|A| = 1 \times 2 = 2$ $\text{adj}A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ $\therefore A^{-1} = \frac{\text{adj}A}{|A|} = \frac{1}{2} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$

(2) $\left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{array} \right) \therefore A^{-1} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$

(c) $(A^T)^{-1} = (A^{-1})^T = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$

3. (Chapter 3: 30 points) Define matrix A and vector \vec{b} by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_3 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

$$x_3 = \frac{\det(A_3(\vec{b}))}{\det A}$$

$$A_3(\vec{b}) = \begin{pmatrix} -2 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\begin{aligned} \det(A_3(\vec{b})) &= 1 \times \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} + 3 \times \begin{vmatrix} -2 & 3 \\ 0 & -2 \end{vmatrix} \\ &= 1 \times (6 + 2) + 3 \times 4 = 8 + 12 = 20 \end{aligned}$$

$$\det A = 1 \times \begin{vmatrix} 3 & 0 \\ -2 & 4 \end{vmatrix} + (-2) \times \begin{vmatrix} -2 & 3 \\ 0 & -2 \end{vmatrix}$$

$$= 1 \times 12 + (-2) \times 4$$

$$= 12 - 8 = 4$$

$$\therefore x_3 = \frac{20}{4} = 5$$

4. (Chapters 1 to 4: 30 points) Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -3 & -2 & -1 \\ -1 & 0 & 0 \\ 6 & 6 & 3 \\ 2 & 2 & 1 \end{pmatrix}$$

- (a) Check the independence tests below which apply to prove that the column vectors of the matrix A are independent in the vector space \mathcal{R}^4 .
- (b) Show the details for one of the independence tests that you checked.

- | | | |
|--------------------------|---------------------------|---|
| <input type="checkbox"/> | Wronskian test | Wronskian of $\vec{f}_1, \vec{f}_2, \vec{f}_3$ nonzero at $x = x_0$ implies independence of $\vec{f}_1, \vec{f}_2, \vec{f}_3$. |
| <input type="checkbox"/> | Rank test | Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3. |
| <input type="checkbox"/> | Determinant test | Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant. |
| <input type="checkbox"/> | Euler Atom test | Any finite set of distinct atoms is independent. |
| <input type="checkbox"/> | Sample test | Functions $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are independent if a sampling matrix has nonzero determinant. |
| <input type="checkbox"/> | Pivot test | Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns. |
| <input type="checkbox"/> | Orthogonality test | A set of nonzero pairwise orthogonal vectors is independent. |
| <input type="checkbox"/> | Combination test | A list of vectors is independent if each vector is not a linear combination of the preceding vectors. |

5. (Chapters 2, 4: 20 points) Define S to be the set of all vectors \vec{x} in \mathcal{R}^3 such that $x_1 + x_3 = x_2$, $x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of \mathcal{R}^3 .

the vector space for all vectors \vec{x} in \mathcal{R}^3 that satisfies three equations are

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + x_2 + x_3 = 0 \\ x_3 = 0 \end{cases}$$

so S is null space of the 3×3 ^{system} \checkmark on the left
Thus S is subspace of \mathcal{R}^3

6. (Chapter 6: 40 points) Let S be the subspace of \mathbb{R}^4 spanned by the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find a Gram-Schmidt orthonormal basis of S .

Let $\vec{w}_1 = \vec{v}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1+1+0+0}{1+1+1+0} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/3 \\ 2/3 \\ 2/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1 \end{pmatrix}$$

normalize \vec{w}_1 \vec{w}_2 to get \vec{u}_1 \vec{u}_2

$$\vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{1}{\sqrt{1+1+1}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{pmatrix}$$

$$\begin{aligned} \vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{1}{\sqrt{(\frac{1}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2 + 1}} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{9}{9}}} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1 \end{pmatrix} \\ &= \frac{3}{\sqrt{15}} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{15}/3 \\ \sqrt{15}/3 \\ \sqrt{15}/3 \\ \sqrt{15} \end{pmatrix} \end{aligned}$$

\vec{u}_1 \vec{u}_2 are orthonormal basis for S

7. (Chapters 1 to 6: 30 points) Let A be an $m \times n$ matrix and assume that $A^T A$ has nonzero determinant. Prove that the rank of A equals n .

Since $A^T A$ has non zero determinant, $A^T A$ is invertible, ie $(A^T A)^{-1}$ exist

Let \vec{x} be a vector which satisfies $A\vec{x} = \vec{0}$ (1)

(since A is $m \times n$ \vec{x} would be $n \times 1$)

let multiply by A^T $A^T A \vec{x} = \vec{0}$ (2)

since $A^T A$ has inverse $(A^T A)^{-1}$ left multiply (2) by $(A^T A)^{-1}$

$$(A^T A)^{-1} A^T A \vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$$

Which means the \vec{x} that satisfies $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

This means $A\vec{x} = \vec{0}$ has only trivial solution, ie all columns in A are pivot columns and $m \geq n$.

Ie there are n pivot columns and thus rank of A is n .

used rank + nullity = # vars = n

$$\text{rank} = n \Leftrightarrow \text{nullity} = 0 \Leftrightarrow \text{Null}(A) = \{\vec{0}\}$$

8. (Chapter 5: 40 points) The matrix A below has eigenvalues 3, 3 and 3. Test A to see if it is diagonalizable, and if it is, then display three eigenpairs of A .

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

A

Try $(A - 3I)x = 0$ and see how many eigen vectors we find

$$\begin{pmatrix} 4-3 & 1 & 1 \\ -1 & 2-3 & 1 \\ 0 & 0 & 3-3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there is only one free variable
in matrix on the left
It has only one eigenvector.

So A has only one eigen pair, A is
not diagonalizable.

9. (Chapter 6: 30 points) Let W be the column space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let

$\vec{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Let $\vec{\tilde{b}}$ be the near point to \vec{b} in the subspace W . Find $\vec{\tilde{b}}$.

A

$$A^T A \vec{x} = A^T \vec{b}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+1+1 & 1+1 \\ 1+1 & 1+1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+(-1)+1 \\ 1+(-1)+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So solve \vec{x} by $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ augmented matrix $\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 2 & 2 & 0 \end{array} \right)$

row reduce augmented matrix $\left(\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 1 & 1 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -1 \end{array} \right)$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array} \right) \therefore \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{\tilde{b}} = A \vec{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

10. (Chapter 6: 30 points) Let Q be an orthogonal matrix with columns $\vec{q}_1, \vec{q}_2, \vec{q}_3$. Let D be a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \lambda_3$. Prove that the 3×3 matrix $A = QDQ^T$ satisfies $A = \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$.

Since $A = QDQ^T$ Q has orthogonal columns $\vec{q}_1, \vec{q}_2, \vec{q}_3$ we know $\vec{q}_1, \vec{q}_2, \vec{q}_3$ are eigenvectors of A
 D be diagonal matrix with entries $\lambda_1, \lambda_2, \lambda_3$

so $A\vec{q}_1 = \lambda_1 \vec{q}_1$

multiply $\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$ by \vec{q}_1 since $\vec{q}_1 \perp \vec{q}_2, \vec{q}_1 \perp \vec{q}_3$

Another proof:

$$Q = \langle \vec{q}_1 | \vec{q}_2 | \vec{q}_3 \rangle \quad Q^T = \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{q}_1 & | & \vec{q}_2 & | & \vec{q}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \vec{q}_1 & | & \lambda_2 \vec{q}_2 & | & \lambda_3 \vec{q}_3 \end{bmatrix} \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \end{bmatrix}$$

$$= \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$$

$$\vec{q}_1 (\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T)$$

$$= \lambda_1 \vec{q}_1 \quad (\vec{q}_1 \text{ is unit vector since it came from orthogonal matrix})$$

in the same way we can get

$$A\vec{q}_2 = \lambda_2 \vec{q}_2 = \vec{q}_2 (\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T)$$

$$A\vec{q}_3 = \lambda_3 \vec{q}_3 = \vec{q}_3 (\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T)$$

so $A = \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$

11. (Chapter 7: 30 points) The spectral theorem says that a symmetric matrix A can be factored into $A = QDQ^T$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$.

A

find eigenvalues of A

$$\begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(4-\lambda) - (-1)(-1) = 0 \quad \textcircled{1} \text{ solve for eigen vector}$$

$$(\lambda-4)(\lambda-4) - 1 = 0$$

$$\lambda^2 - 8\lambda + 16 - 1 = 0$$

$$\lambda^2 - 8\lambda + 15 = 0$$

$$(\lambda-3)(\lambda-5) = 0$$

$$\therefore \lambda_1 = 5, \lambda_2 = 3$$

$$\begin{pmatrix} 4-5 & -1 \\ -1 & 4-5 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \therefore \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

② solve for the second eigen vector

$$\begin{pmatrix} 4-3 & -1 \\ -1 & 4-3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\text{normalize } \vec{v}_1 \text{ get } \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\therefore \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{normalize } \vec{v} \text{ get } \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\text{So } D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \quad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad Q^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

12. (Chapter 7: 30 points) Write out the singular value decomposition for the matrix

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4+1 & 4-1 \\ 4-1 & 4+1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \quad \wedge$$

Try solve for eigen values of $A^T A$ $\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$

$$(5-\lambda)(5-\lambda) - 9 = 0$$

$$(\lambda-5)(\lambda-5) - 9 = 0$$

$$\lambda^2 - 10\lambda + 25 - 9 = 0$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda-2)(\lambda-8) = 0$$

$$\therefore \lambda_1 = 8 \quad \lambda_2 = 2$$

So singular values for A are

$$\sigma_1 = \sqrt{8} = 2\sqrt{2}$$

$$\sigma_2 = \sqrt{2}$$

Solve for eigen vectors of $A^T A$

$$\begin{pmatrix} 5-8 & 3 \\ 3 & 5-8 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \quad \text{unitize to get } \vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Solve for the other eigen vector

$$\begin{pmatrix} 5-2 & 3 \\ 3 & 5-2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \quad \text{unitize } \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{aligned} \vec{u}_1 &= \frac{A\vec{v}_1}{\sigma_1} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{1}{2\sqrt{2}} \begin{pmatrix} \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2\sqrt{2}} \cdot \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

$$\vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

ii SVD for A is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

13. (Chapter 4: 30 points) Let the linear transformation T from \mathcal{R}^3 to \mathcal{R}^3 be defined by its action on three independent vectors:

$$T \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}, T \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}, T \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}. \quad A$$

Find the unique 3×3 matrix A such that T is defined by the matrix multiply equation $T(\vec{x}) = A\vec{x}$.

Write the three transformations as one, let T work on each column of the matrix on right of dot product form

$$A \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 4 \\ 4 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix} \quad \text{Let } B \text{ represent } \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

since determinant of B is not 0 $|B| = 1 \times (-1) \times 4 + 1 \times 1 \times 6 = 2$

$$\therefore B^{-1} \text{ exist } A = \begin{pmatrix} 4 & 5 & 4 \\ 4 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix} \times B^{-1}$$

Solve for B^{-1} first

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} & 1 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & 3 \end{array} \right) \therefore B^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & -1 \\ -1 & \frac{1}{2} & -2 \\ 1 & -\frac{1}{2} & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 5 & 4 \\ 4 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & -1 \\ -1 & \frac{1}{2} & -2 \\ 1 & -\frac{1}{2} & 3 \end{pmatrix} = \begin{pmatrix} 0 \times 4 + 5 \times (-1) + 4 \times 1 & \frac{1}{2} \times 4 + \frac{3}{2} \times 5 + \frac{3}{2} \times 4 & 4 \times (-1) + 5 \times (-2) + 4 \times 3 \\ 0 \times 4 + 1 \times (-1) + 0 & \frac{1}{2} \times 4 + \frac{3}{2} \times 1 & 4 \times (-1) - 2 + 3 \times 4 \\ 2 \times 0 + 1 \times (-1) + 2 \times 1 & \frac{1}{2} \times 2 + \frac{3}{2} \times 1 - \frac{3}{2} \times 2 & -2 - 2 + 6 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \frac{7}{2} & -2 \\ -1 & \frac{7}{2} & -6 \\ 1 & -\frac{1}{2} & 2 \end{pmatrix}$$

14. (Chapter 4, 7: 40 points) Let A be an $m \times n$ matrix. Denote by S_1 the row space of A and S_2 the column space of A . It is known that S_1 and S_2 have dimension $r = \text{rank}(A)$. Let $\vec{p}_1, \dots, \vec{p}_r$ be a basis for S_1 and let $\vec{q}_1, \dots, \vec{q}_r$ be a basis for S_2 . For example, select the pivot columns of A^T and A , respectively. Define $T : S_1 \rightarrow S_2$ initially by $T(\vec{p}_i) = \vec{q}_i$, $i = 1, \dots, r$. Extend T to all of S_1 by linearity, which means the final definition is

$$T(c_1\vec{p}_1 + \dots + c_r\vec{p}_r) = c_1\vec{q}_1 + \dots + c_r\vec{q}_r.$$

A or A^T

Prove that T is one-to-one and onto.

one to one:

Let \vec{x} be a vector in S_1 , i.e. row space of A

Let \vec{x}_1, \vec{x}_2 be two vectors that are equal after linear transformation T , and $\vec{x} = \vec{x}_1$

$$\text{i.e. } A\vec{x}_1 = A\vec{x}_2 \Rightarrow A\vec{x}_1 - A\vec{x}_2 = \vec{0} \quad A(\vec{x}_1 - \vec{x}_2) = \vec{0} \quad A\vec{x} = \vec{0}$$

i.e. \vec{x} in nullspace of A since nullspace of $A \perp$ row-space of A and intersection is $\vec{0}$

so $\vec{x} = \vec{0}$ i.e. $\vec{x}_1 = \vec{x}_2$ so one-to-one proven

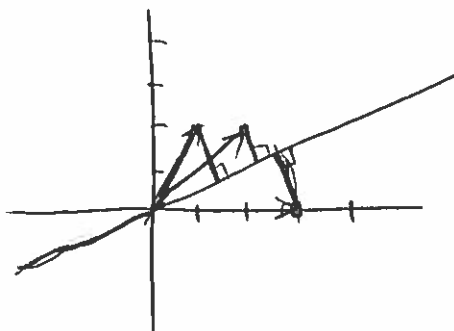
onto: Let \vec{y} be any vector in S_2 , i.e. column space of A , i.e. \vec{y} is linear combination of columns of A . So there exist a \vec{x} whose entries are weights of linear combination that $A\vec{x} = \vec{y}$ so onto-proven.

A better paragraph would do much more. The coefficients from the l.c. of cols have to be applied to basis $\{\vec{p}_i\}$ to define \vec{x} and that $T(\vec{x}) = \vec{y}$.

15. (Chapter 4: 20 points) Least squares can be used to find the best fit line for the points $(1, 2)$, $(2, 2)$, $(3, 0)$. Without finding the line equation, describe how to do it, in a few sentences.

the way to find such a line is to find one with the least error combined

D+



16. (Chapters 1 to 7: 20 points) State the Fundamental Theorem of Linear Algebra. Include Part 1: The dimensions of the four subspaces, and Part 2: The orthogonality equations for the four subspaces.

Given matrix A , there are four spaces of A : $\text{col } A$, $\text{Nul } A$, $\text{row space of } A$, $\text{Nul } A^T$.

part two: $\text{Nul } A \perp$ to row space of A

Substitute A with A^T in above

$$\text{Nul } A^T \perp \text{col } A$$

Let r be rank of A . A be $m \times n$

part 1 row space of A has dimension r

$$\dim(\text{col } A) = r$$

$$\text{Nul } A = n - r$$

$$\text{Nul } A^T = m - r$$

17. (Chapter 7: 20 points) State the Spectral Theorem for symmetric matrices. Include the important results included in the spectral theorem, about real eigenvalues and diagonalizability. Then discuss the spectral decomposition.

A or A^T

Eigenvalues are sometimes called Spectral Spectrum

- (1) Eigen vectors of symmetric matrix that correspond to distinct eigen values are orthogonal with each other
- (2) Dimension of eigen spaces of symmetric matrix are the multiplicity of corresponding eigen values. Also eigen space that correspond to distinct eigen values are orthogonal to each other.
- (3) Symmetric matrices are all diagonalizable.
- (4) If all eigen values are > 0 then A is positive definite
If all eigen values are < 0 then A is negative definite
If A has both positive and negative eigen values, then A is indefinite
- (5) D is diagonal with entries coming from A 's eigen values
 P is orthogonal with columns being A 's eigen vector after being united
then $A = PDPT^T = PDPT^T$

yes,
but Gram-Schmidt
is used here also.