MATH 2270-2 Final Exam Spring 2016

NAME (please print): Jie Zhang u1013501

QUESTION	VALUE	SCORE	
1	60	57	−3
2	40		A
3	30		A
4	30		
5	20		A
6	40		A
7	30		A
8	40		A
9	30		A
10	30		A
11	30		A
12	30		A
13	30		A
14	40		A or A− −1
15	20		D+ −8
16	20		A
17	20		A− or A −1
TOTAL	540	500	−10

98.0%

No books, notes or electronic devices, please.

The questions have credits which reflect the time required to write the solution.

If you must write a solution out of order or on the back side, then supply a road map.

Solutions are expected to include readable and convincing details. A correct answer without details earns 25%.

Expect about 3 to 10 minutes per problem. Final exam problems may have multiple parts.

1. (Chapter 1: 60 points) Consider the system $A\vec{u} = \vec{b}$ with $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ defined by

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + x_4 &= 2 \\ 4x_1 + 3x_2 + 8x_3 + x_4 &= 4 \\ 6x_1 + 3x_2 + 8x_3 + x_4 &= 2 \end{aligned}$$

Solve the following parts:

A **(a)** [10%] Find the reduced row echelon form of the augmented matrix.

A **(b)** [10%] Identify the **free** variables and the **lead** variables.

−3 C+ **(c)** [10%] Display a vector formula for a particular solution $\vec{u}_p$.

would be correct

(d) [10%] Display a vector formula for the homogeneous solution $\vec{u}_h$.

(e) [10%] Identify each of **Strang's Special Solutions.**

(f) [10%] Display the vector general solution $\vec{u}$, using **superposition.**

0 −3 0 −1 0
0 3 2 1 2

Augmented matrix:

$$\left(\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 2 \\ 4 & 3 & 8 & 1 & 4 \\ 6 & 3 & 8 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 1 \\ 0 & -3 & 0 & -1 & 0 \\ 0 & -6 & -4 & -2 & -4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 3 & 2 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 2 & 0 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

→ reduced row echelon form

lead variable: $x_1\ x_2\ x_3$ free variable x_4

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{3} \\ 1 \\ x_4\text{free} \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So particular solution $\vec{u}_p = \begin{bmatrix} -1 \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$

homogeneous solution $\vec{u}_h = t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $-\infty < t < \infty$

Strang's special solution $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

vector general solution $\begin{bmatrix} -1 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$ $-\infty < t < \infty$

2. (Chapter 2: 40 points)

(a) [10%] Describe for $n \times n$ matrices two different methods for finding the matrix inverse.

(b) [20%] Apply the two methods to find the inverse of the matrix $A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}$.

(c) [10%] Find the inverse of the transpose of the matrix in part (b).

A

1(a) (1) using $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$

(2) using $(A|I) \sim (I|A^{-1})$

(b) (1) $|A| = 1 \times 2 = 2$ $\operatorname{adj} A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ $\therefore A^{-1} = \dfrac{\operatorname{adj}}{|A|} = \frac{1}{2}\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$

(2) $\left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right) \therefore A^{-1} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$

(c) $(A^T)^{-1} = (A^{-1})^T = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$

3. (Chapter 3: 30 points) Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

A

Find the value of x_3 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

$$x_3 = \frac{\det(A_3(\vec{b}))}{\det A}$$

$$A_3(\vec{b}) = \begin{pmatrix} -2 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\det(A_3(\vec{b})) = 1 \times \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} + 3 \times \begin{vmatrix} -2 & 3 \\ 0 & -2 \end{vmatrix}$$

$$= 1 \times (6+2) + 3 \times 4 = 8 + 12 = 20$$

$$\det A = 1 \times \begin{vmatrix} 3 & 0 \\ -2 & 4 \end{vmatrix} + (-2) \begin{vmatrix} -2 & 3 \\ 0 & -2 \end{vmatrix}$$

$$= 1 \times 12 + (-2) \times 4$$

$$= 12 - 8 = 4$$

$$\therefore x_3 = \frac{20}{4} = 5$$

4. (Chapters 1 to 4: 30 points) Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -3 & -2 & -1 \\ -1 & 0 & 0 \\ 6 & 6 & 3 \\ 2 & 2 & 1 \end{pmatrix}$$

(a) Check the independence tests below which apply to prove that the column vectors of the matrix A are independent in the vector space $\mathcal{R}^4$.
(b) Show the details for one of the independence tests that you checked.

☐	**Wronskian test**	Wronskian of $\vec{f}_1, \vec{f}_2, \vec{f}_3$ nonzero at $x = x_0$ implies independence of $\vec{f}_1, \vec{f}_2, \vec{f}_3$.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Euler Atom test**	Any finite set of distinct atoms is independent.
☐	**Sample test**	Functions $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are independent if a sampling matrix has nonzero determinant.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Orthogonality test**	A set of nonzero pairwise orthogonal vectors is independent.
☐	**Combination test**	A list of vectors is independent if each vector is not a linear combination of the preceding vectors.

5. **(Chapters 2, 4: 20 points)** Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = x_2$, $x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

the vector space for all vectors $\vec{x}$ in R^3 that satisfies three equations are

A

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + x_2 + x_3 = 0 \\ x_3 = 0 \end{cases}$$

so S is null space of the 3x3 system on the left

Thus S is subspace of R^3

6. (Chapter 6: 40 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find a Gram-Schmidt orthonormal basis of S.

A

Let $\vec{w}_1 = \vec{v}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \vec{v}_1}{\vec{v}_1 \vec{v}_1}\vec{v}_1$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1+1+0+0}{1+1+1+0}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-\frac{2}{3} \\ 1-\frac{2}{3} \\ 0-\frac{2}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix}$$

unitize $\vec{w}_1$ $\vec{w}_2$ to get $\vec{u}_1$ $\vec{u}_2$

$$\vec{u}_1 = \frac{\vec{w}_1}{|\vec{u}_1|} = \frac{1}{\sqrt{1+1+1}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{|\vec{u}_2|} = \frac{1}{\sqrt{(\frac{1}{3})^2 + (\frac{1}{3})^2 + (-\frac{2}{3})^2 + 1}}\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{9}{9}}}\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix}$$

$$= \frac{3}{\sqrt{15}}\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ \frac{-2}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{pmatrix}$$

$\vec{u}_1$ $\vec{u}_2$ are orthonormal basis for S

7. (Chapters 1 to 6: 30 points) Let A be an $m \times n$ matrix and assume that $A^T A$ has nonzero determinant. Prove that the rank of A equals n.

A

Since A^TA has non zero determinant, A^TA is invertible, ie $(A^TA)^{-1}$ exist

Let $\vec{x}$ be a vector which satisfies $A\vec{x} = \vec{0}$ (1)

(Since A is $m \times n$ $\vec{x}$ would be $n \times 1$)

left multiply by A^T $A^TA\vec{x} = \vec{0}$ (2)

Since A^TA has inverse $(A^TA)^{-1}$ left multiply (2) by $(A^TA)^{-1}$

$(A^TA)^{-1} A^TA\vec{x} = 0 \Rightarrow \vec{x} = 0$

Which means the $\vec{x}$ that satisfies $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

This means $A\vec{x} = \vec{0}$ has only trivial solution, ie all columns in A are pivot columns and $m \geq n$.

Ie there are n pivot colums and thus rank of A is n.

Used rank + nullity = # vars = n

rank $= n \Leftrightarrow$ nullity $= 0 \Leftrightarrow \text{Null}(A) = \{\vec{0}\}$

8. (Chapter 5: 40 points) The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display three eigenpairs of A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

A

Try $(A-3I)\cancel{x}=0$ and see how many eigenvectors we find

$$\begin{pmatrix} 4-3 & 1 & 1 \\ -1 & 2-3 & 1 \\ 0 & 0 & 3-3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there is only one free variable in matrix on the left
It has only one eigenvector.
So A has only one eigenpair, A is not diagonalizable.

9. (Chapter 6: 30 points) Let W be the column space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Let $\hat{\vec{\mathbf{b}}}$ be the near point to $\vec{\mathbf{b}}$ in the subspace W. Find $\hat{\vec{\mathbf{b}}}$.

A

$$A^TA\hat{\vec{x}} = A^T\vec{b}$$

$$A^TA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+1+1 & 1+1 \\ 1+1 & 1+1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^T\vec{b} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+(-1)+1 \\ 1+(-1)+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So solve $\hat{\vec{x}}$ by $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ augmented matrix $\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 2 & 2 & 0 \end{array}\right)$

row reduce augmented matrix $\left(\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -1 \end{array}\right)$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right) \therefore \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\hat{\vec{b}} = A\hat{\vec{x}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

10. (Chapter 6: 30 points) Let Q be an orthogonal matrix with columns $\vec{q}_1, \vec{q}_2, \vec{q}_3$. Let D be a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \lambda_3$. Prove that the 3×3 matrix $A = QDQ^T$ satisfies $A = \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$.

A

Since $A = QDQ^T$ Q has orthogonal columns $\vec{q}_1\ \vec{q}_2\ \vec{q}_3$ we know $\vec{q}_1\ \vec{q}_2\ \vec{q}_3$ are
D be diagonal matrix with entries $\lambda_1, \lambda_2 \lambda_3$
eigen vectors of A

so $A\vec{q}_1 = \lambda_1 \vec{q}_1$

multiply $\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$ by $\vec{q}_1$ since $\vec{q}_1 \perp \vec{q}_2$, $\vec{q}_1 \perp \vec{q}_3$

$\vec{q}_1 (\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$

$= \lambda_1 \vec{q}_1$ ($\vec{q}_1$ is unit vector since it came from orthogonal matrix)

in the same way we can get

$A\vec{q}_2 = \lambda_2 \vec{q}_2 = \vec{q}_2(\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T +$

$A\vec{q}_3 = \lambda_3 \vec{q}_3 = \vec{q}_3(\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3$

so $A = \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3$

Another proof:

$Q = \langle \vec{q}_1 | \vec{q}_2 | \vec{q}_3 \rangle \quad Q^T = \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \end{bmatrix}$

$$A = \left[\vec{q}_1 \middle| \vec{q}_2 \middle| \vec{q}_3 \right] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \end{bmatrix}$$

$$= \left[\lambda_1 \vec{q}_1 \middle| \lambda_2 \vec{q}_2 \middle| \lambda_3 \vec{q}_3 \right] \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \end{bmatrix}$$

$$= \lambda \vec{q}_1 \cdot \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$$

11. (Chapter 7: 30 points) The spectral theorem says that a symmetric matrix A can be factored into $A = QDQ^T$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$.

A

find eigenvalues of A

$$\begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(4-\lambda)-(-1)(-1)=0$$
$$(\lambda-4)(\lambda-4)-1=0$$
$$\lambda^2-8\lambda+16-1=0$$
$$\lambda^2-8\lambda+15=0$$
$$(\lambda-3)(\lambda-5)=0$$
$$\therefore \lambda_1=5, \lambda_2=3$$

① solve for eigenvector

$$\begin{pmatrix} 4-5 & -1 \\ -1 & 4-5 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \therefore \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

unitize $\vec{v}_1$ get

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

② solve for the second eigenvector

$$\begin{pmatrix} 4-3 & -1 \\ -1 & 4-3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

unitize $\vec{v}$ get

$$\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

So $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ $Q = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ $Q^T = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

12. (Chapter 7: 30 points) Write out the singular value decomposition for the matrix $A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$.

$$A^TA = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4+1 & 4-1 \\ 4-1 & 4+1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

Try solve for eigen values of A^TA $\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$

$$(5-\lambda)(5-\lambda) - 9 = 0$$
$$(\lambda-5)(\lambda-5) - 9 = 0$$
$$\lambda^2 - 10\lambda + 25 - 9 = 0$$
$$\lambda^2 - 10\lambda + 16 = 0$$
$$(\lambda-2)(\lambda-8) = 0$$
$$\therefore \lambda_1 = 8 \quad \lambda_2 = 2$$

So singular values for **A** are

$\sigma_1 = \sqrt{8} = 2\sqrt{2}$

$\sigma_2 = \sqrt{2}$

Solve for eigen vectors of A^TA

$$\begin{pmatrix} 5-8 & 3 \\ 3 & 5-8 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$$

unitize to get $\vec{v}_1$

So $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Solve for the other eigen vector

$$\begin{pmatrix} 5-2 & 3 \\ 3 & 5-2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

unitize

$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{1}{\sigma_1}\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2\sqrt{2}} \cdot \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\therefore$ SVD for A is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

13. (Chapter 4: 30 points) Let the linear transformation T from $\mathcal{R}^3$ to $\mathcal{R}^3$ be defined by its action on three independent vectors:

$$T\left(\begin{pmatrix}3\\2\\0\end{pmatrix}\right)=\begin{pmatrix}4\\4\\2\end{pmatrix},\ T\left(\begin{pmatrix}0\\2\\1\end{pmatrix}\right)=\begin{pmatrix}5\\1\\1\end{pmatrix},\ T\left(\begin{pmatrix}1\\2\\1\end{pmatrix}\right)=\begin{pmatrix}4\\0\\2\end{pmatrix}.$$

A

Find the unique 3×3 matrix A such that T is defined by the matrix multiply equation $T(\vec{x})=A\vec{x}$.

Write the three transformations into one, ie T work on each column of the matrix on right of dot product form

$$A\begin{pmatrix}3&0&1\\2&2&2\\0&1&1\end{pmatrix}=\begin{pmatrix}4&5&4\\4&1&0\\2&1&2\end{pmatrix}\quad \text{Let B represent } \begin{pmatrix}3&0&1\\2&2&2\\0&1&1\end{pmatrix}$$

Since determinant of B is not 0 $|B| = 1\times(-1)\times 4 + 1\times 1\times 6 = 2$

$\therefore B^{-1}$ exist $A=\begin{pmatrix}4&5&4\\4&1&0\\2&1&2\end{pmatrix}\times B^{-1}$

Solve for B^{-1} first

$$\left(\begin{array}{ccc|ccc}3&0&1&1&0&0\\2&2&2&0&1&0\\0&1&1&0&0&1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&\frac{1}{3}&\frac{1}{3}&0&0\\1&1&1&0&\frac{1}{2}&0\\0&1&1&0&0&1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&\frac{1}{3}&\frac{1}{3}&0&0\\0&1&\frac{2}{3}&-\frac{1}{3}&\frac{1}{2}&0\\0&1&1&0&0&1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}1&0&\frac{1}{3}&\frac{1}{3}&0&0\\0&1&\frac{2}{3}&-\frac{1}{3}&\frac{1}{2}&0\\0&0&\frac{1}{3}&\frac{1}{3}&-\frac{1}{2}&1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&0&0&\frac{1}{2}&-1\\0&1&0&-1&\frac{3}{2}&-2\\0&0&1&1&-\frac{3}{2}&3\end{array}\right)\quad\therefore B^{-1}=\begin{pmatrix}0&\frac{1}{2}&-1\\-1&\frac{3}{2}&-2\\1&-\frac{3}{2}&3\end{pmatrix}$$

$$A=\begin{pmatrix}4&5&4\\4&1&0\\2&1&2\end{pmatrix}\begin{pmatrix}0&\frac{1}{2}&-1\\-1&\frac{3}{2}&-2\\1&-\frac{3}{2}&3\end{pmatrix}=\begin{pmatrix}0\times4+5\times(-1)+4\times1 & \frac{1}{2}\times4+\frac{3}{2}\times5+(\frac{-3}{2})\times4 & 4\times(-1)+5(\ +3\times4\\ 0\times4+1\times(-1)+0 & \frac{1}{2}\times4+\frac{3}{2}\times1 & 4\times(-1)-2\\ 2\times0+1\times(-1)+2\times1 & \frac{1}{2}\times2+\frac{3}{2}\times1-\frac{3}{2}\times2 & -2-2+6\end{pmatrix}$$

$$=\begin{pmatrix}-1&\frac{7}{2}&-2\\-1&\frac{7}{2}&-6\\1&-\frac{1}{2}&2\end{pmatrix}$$

14. (Chapter 4, 7: 40 points) Let A be an $m \times n$ matrix. Denote by S_1 the row space of of A and S_2 the column space of A. It is known that S_1 and S_2 have dimension $r = \mathbf{rank}(A)$. Let $\vec{p}_1, \ldots, \vec{p}_r$ be a basis for S_1 and let $\vec{q}_1, \ldots, \vec{q}_r$ be a basis for S_2. For example, select the pivot columns of A^T and A, respectively. Define $T : S_1 \to S_2$ initially by $T(\vec{p}_i) = \vec{q}_i, \quad i = 1, \ldots, r$. Extend T to all of S_1 by linearity, which means the final definition is

A or A−

$$T(c_1\vec{p}_1 + \cdots + c_r\vec{p}_r) = c_1\vec{q}_1 + \cdots + c_r\vec{q}_r.$$

Prove that T is one-to-one and onto.

one to one:

Let $\vec{x}$ be a vector in S_1, ie row space of A

Let $\vec{x}_1$ $\vec{x}_2$ be two vectors that are equal after linear tranformatis T, and $\vec{x} = \vec{x}_1$

ie $A\vec{x}_1 = A\vec{x}_2 \Rightarrow A\vec{x}_1 - A\vec{x}_2 = 0$ $\quad A(\vec{x}_1 - \vec{x}_2) = 0$ $\quad A\vec{x} = \vec{0}$

ie $\vec{x}$ in nullspace of A since nullspace of $A \perp$ row-space of A and intersection

so $\vec{x} = \vec{0}$ ie $\vec{x}_1 = \vec{x}_2$ so one to one proven

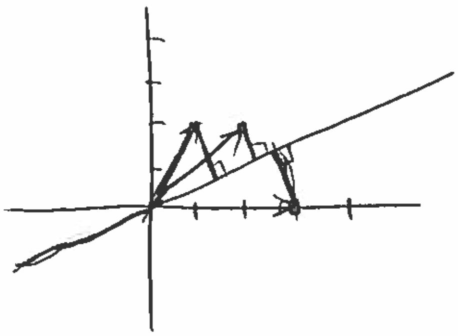
Onto: Let $\vec{y}$ be any vector in S_2, ie column space of A, ie $\vec{y}$ is linear combination of columns of A so there exist a $\vec{x}$ whose entries are weight of linear combination that $A\vec{x} = \vec{y}$ so onto proven

A better paragraph would do much more. The coefficients from the l.c. of Cols have to be applied to basis $\{\vec{p}_i\}$ to define $\vec{x}$ such that $T(\vec{x}) = \vec{y}$.

15. (Chapter 4: 20 points) Least squares can be used to find the best fit line for the points $(1,2)$, $(2,2)$, $(3,0)$. Without finding the line equation, describe how to do it, in a few sentences.

the way to find such a line is to find one with the least error combined

D+



16. (Chapters 1 to 7: 20 points) State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

A

Given matrix A, there are four spaces of A Col A . Nul A row space of A .
Nul A^T

part two: Nul A $\perp$ to row space of A

Substitute A with A^T in above

Nul $A^T \perp$ Col A

Let r be rank of A . A be m×n

part 1 row space of A has dimension r

$\dim(\text{Col } A) = r$

Nul $A = n-r$

Nul $A^T = m-r$

17. (Chapter 7: 20 points) State the **Spectral Theorem** for symmetric matrices. Include the important results included in the spectral theorem, about real eigenvalues and diagonalizability. Then discuss the **spectral decomposition**.

A− or A

Eigenvalues are sometimes called ~~Spectral~~ Spectrum

(1) Eigen vectors of symmetric matrix that correspond to distinct eigen values are orthogonal with each other

(2) dimension of eigen spaces of symmetric matrix are the multiplicity of corresponding eigen values. Also eigenspace that correspond to distinct eigen values are orthogonal to each other.

(3) Symmetric matrices are all diagonalizable

(4) If all eigen values are >0 the A is positive definite

If all eigenvalues are <0 then A is negative definite

If A has both positive and negative eigen values, then A is indefinite

(5) D is diagonal with entries coming from A's eigenvalues

P is orthogonal with columns being A's eigen vector after being unized

then $A = PDP^T = PDP^{-1}$

yes, but Gram-Schmidt is used here also.