Sample Quiz2   S2014 Math 2250

Problem 1

ANswers: (1) $u = 68 + 117 e^{-ht}$

(2) $h = (-1/50) \ln\left(\frac{113}{117}\right) \cong 0.0006957$

(3) $t \cong 345.52$ seconds, about 6 min.

Details (1). Because $u =$ degrees F and $t =$ seconds, then the model is $\begin{cases} u' = -h(u - 68), \\ u(0) = 185, \quad u(50) = 181. \end{cases}$

The DE is solved by superposition $u = u_h + u_p$. The equilibrium solution is $u_p = 68$. Then $u_h = \frac{c}{\text{integ. factor}} = c e^{-ht}$, using standard linear form $u' + hu = 68h$. condition $u(0) = 185$ is used on $u = u_h + u_p = c e^{-ht} + 68$ to evaluate $185 = c e^0 + 68$, Then $c = 185 - 68 = 117$.

Details (2). Start with answer (1) and use $u(50) = 181$.
Then
$$181 = 68 + 117 e^{-ht} \qquad \text{when } t = 50$$
$$113 = 117 e^{-ht}$$
$$e^{-ht} = \frac{113}{117} \Rightarrow -ht = \ln\left(\frac{113}{117}\right)$$
$$\Rightarrow h = \frac{-1}{50} \ln\left(\frac{113}{117}\right)$$

Details (3). As in details (2),
$$160 = 68 + 117 e^{-ht}$$
$$e^{-ht} = \frac{160 - 68}{117}$$
$$-ht = \ln\left(\frac{92}{117}\right) \qquad \text{take log across eqtn.}$$
$$t = \left(\frac{-1}{h}\right) \ln\left(\frac{92}{117}\right)$$
$$t = 50 \, \frac{\ln(92/117)}{\ln(113/117)} \cong 345.52 \text{ seconds}$$

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Problem 2

Answers   (1) $x' = 0.8x(1 - \frac{x}{780500})$, $x = 0, 780500$

(2) $x^2 - kx + \frac{HK}{r} = 0$, roots $= \frac{K}{2} \pm \frac{1}{2}\sqrt{k^2 - \frac{4HK}{r}}$
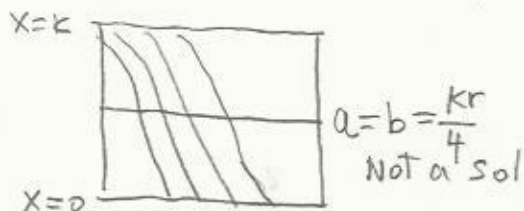
(3)

$a = \frac{K}{2} + \frac{1}{2}\sqrt{D}$

$b = \frac{K}{2} - \frac{1}{2}\sqrt{D}$

$D = k^2 - \frac{4HK}{r}$

Choose $H = \frac{kr}{4} - 5000$

$a = b = \frac{kr}{4}$

Not a sol

Choose $H = \frac{kr}{4} + 5000$
All sols decrease
to zero (and beyond),
meaning extinction

Details (1). $x' = rx(1 - \frac{x}{k})$, Substitute $r = 0.8$, $k = 780500$

Details (2) $x' = rx(1 - \frac{x}{k}) - H = rx - \frac{r}{k}x^2 - H$

$x' = -\frac{r}{k}(x^2 - kx + \frac{HK}{r})$

Apply quadratic formula to $x^2 - kx + \frac{HK}{r} = 0$ to find roots,
reported in both (2), (3) above.

Details (3). A double real root is when the discriminant
$D = k^2 - \frac{4HK}{r} = 0$, requiring $H = \frac{kr}{4}$. For $H < \frac{kr}{4}$,
there are 2 real roots a, b as given in the answer. For
$H > \frac{kr}{4}$ there are no real roots, therefore $x' < 0$ and x
decreases to zero (extinction) and beyond.

The larger root $a = \frac{K}{2} + \frac{1}{2}\sqrt{D}$ is a stable funnel.
The smaller root $b = \frac{K}{2} - \frac{1}{2}\sqrt{D}$ is an unstable spout.
When $D = 0$, then $a = b = \frac{k}{2}$ is a node, when a, b
are real, extinction for $x < b$, carrying capacity $= a$,
sustainable harvest for $x > b$.

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> F:=x->r*x*(1-x/K):G:=x->r*x*(1-x/K)-H:
> solve(G(x)=0,x);
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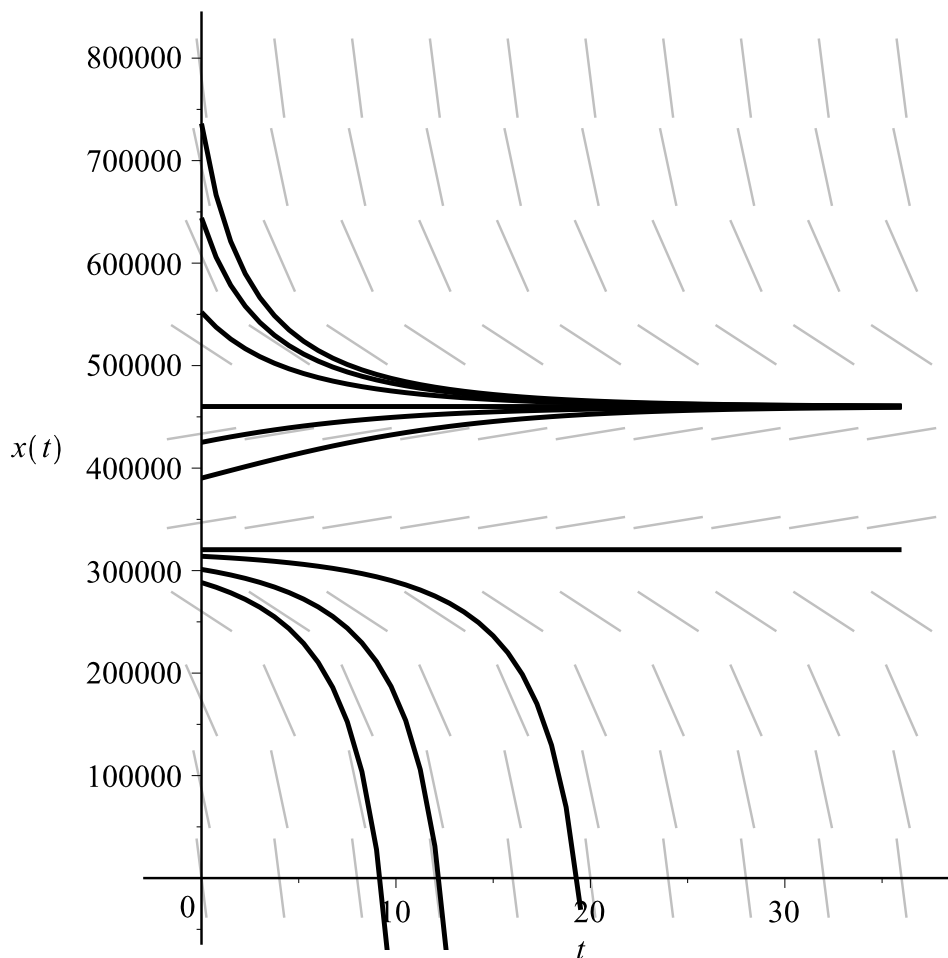
$$\frac{1}{2}\,\frac{Kr+\sqrt{K^2r^2-4HKr}}{r},\ \frac{1}{2}\,\frac{Kr-\sqrt{K^2r^2-4HKr}}{r}$$

```
> de:=diff(x(t),t)=G(x(t)):r:=0.8:K:=780500:H0:=K*r/4:H:=H0-5000:
  a:=K/2+(1/2)*sqrt(K^2-4*H*K/r);b:=K/2-(1/2)*sqrt(K^2-4*H*K/r);
  ic:=[[0,0.9*b],[0,0.94*b],[0,0.98*b],[0,b],[0,(a+b)/2],[0,(3*a+b)
  /4],
  [0,a],[0,1.2*a],[0,1.4*a],[0,1.6*a]]:
  opts:=dirfield=[10,10],arrows=line,color=gray,linecolor=black,
  thickness=2:
> DEtools[DEplot](de,x(t),t=0..36,x=0..K,ic,opts);
```

$$a := 4.600935752\ 10^5$$
$$b := 3.204064248\ 10^5$$


```
> de:=diff(x(t),t)=G(x(t)):
  r:=0.8:K:=780500:H0:=K*r/4:
  H:=H0+6000;a:=K/2;b:=K/2;
  ic:=[[0,0.9*b],[0,0.94*b],[0,0.98*b],[0,b],[0,a],[0,1.2*a],[0,
  1.4*a],
  [0,1.6*a],[0,1.8*a],[0,2*a]]:
```

$$H := 1.621000000 \; 10^5$$

$$a := 390250$$

$$b := 390250$$