Sample Quiz 5 Solutions

Problem 1. Balancing chemical Equations

(a) $\begin{cases} x_1 = x_3 \\ 2x_2 = 2x_4 \\ 2x_1 = x_4 \end{cases} \longrightarrow \begin{cases} x_1 - x_3 = 0 \\ 2x_2 - 2x_4 = 0 \\ 2x_1 - x_4 = 0 \end{cases} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 \\ 2 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(b) $\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 2 & 0 & 0 & -1 & 0 \end{array}\right)$ = augmented matrix

$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 & 0 \end{array}\right)$ mult(2, 1/2)

$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 \end{array}\right)$ combo(1,3,−2)

(c) $\boxed{\begin{array}{l} \text{lead} = x_1, x_2, x_3 \\ \text{free} = x_4 \end{array}}$

$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\tfrac{1}{2} & 0 \end{array}\right)$ mult(3, ½)

$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -\tfrac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\tfrac{1}{2} & 0 \end{array}\right)$ combo(3,1,1)

$\boxed{\text{RREF Found}}$

DEF: RREF = reduced row echelon form
→ Means the equations pass the Last Frame Test

(c) See right column above

(d) Last frame test passed.

$\begin{cases} x_1 - \tfrac{1}{2}x_4 = 0 \\ x_2 - x_4 = 0 \\ x_3 - \tfrac{1}{2}x_4 = 0 \end{cases}$ Last Frame written as equations. Apply Last Frame Algorithm,

$\begin{cases} x_1 = \tfrac{1}{2}x_4 \\ x_2 = x_4 \\ x_3 = \tfrac{1}{2}x_4 \end{cases}$ isolate lead variables left

$\{\ x_4 = t_1$ Assign invented symbols $t_1, t_2, \ldots$ to the free variables, then back-sub

$\begin{cases} x_1 = \tfrac{1}{2}t_1 \\ x_2 = t_1 \\ x_3 = \tfrac{1}{2}t_1 \\ x_4 = t_1 \end{cases}$ Final Answer, must be in <u>variable list order</u>, all variables listed. Only invented symbols on the right.

Sample Quiz 5 Solutions

problem 2. Solving Higher order Initial Value Problems with Linear Algebra

(a) Let $v = y''$ in $y''' + 4y'' = 0$ to get $v' + 4v = 0$. Then $v = \dfrac{c}{\text{integ. factor}} = \dfrac{c}{e^{4x}} = ce^{-4x}$. This means $y'' = ce^{-4x}$. Integrate twice: $y' = c_1 + ce^{-4x}/(-4)$, $y = c_2 + c_1 x + ce^{-4x}/(-4)^2$. Rename the constants to obtain $y = c_1 + c_2 x + c_3 e^{-4x}$

(b) $y(0) = 1$: $\quad c_1 + c_2(0) + c_3 e^0 = 1$

$y'(0) = 2$: $\quad 0c_1 + 1 \cdot c_2 + (-4)c_3 e^0 = 2$

$y''(0) = -1$: $\quad 0c_1 + 0c_2 + (-4)^2 c_3 e^0 = -1$

Then

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 16 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(c) Augmented matrix $= \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 16 & -1 \end{array}\right)$

unique solution case, because lead $= c_1, c_2, c_3$, No free.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & -1/16 \end{array}\right) \quad \text{mult}(3, 1/16)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 17/16 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & -1/16 \end{array}\right) \quad \text{combo}(3, 1, -1)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 17/16 \\ 0 & 1 & 0 & 7/4 \\ 0 & 0 & 1 & -1/16 \end{array}\right) \quad \begin{array}{l}\text{combo}(3, 2, 4) \\ \text{RREF Found}\end{array}$$

$$\begin{cases} c_1 = 17/16 \\ c_2 = 7/4 \\ c_3 = -1/16 \end{cases} \Longrightarrow y(x) = \frac{17}{16} + \frac{7x}{4} - \frac{1}{16}e^{-4x}$$

ANS CHECK: Test $y(0) = 1$, $y'(0) = 2$, $y''(0) = -1$
Passed ✓

Sample Quiz 5 Solutions

problem 3. RL-circuit with DC voltage source

(a) Explain IC. Because $I = Q'$, Then charge Q and current I initially zero means $Q(0)=0$, $I(0)=Q'(0)=0$. Model $LI' + RI = V_s$ implies $LI'(0) + RI(0) = V_s$ or $I'(0) = V_s/L$ ($I(0)=0$). Then $Q''(0) = I'(0) = V_s/L$.

(b) By problem 2, model $LQ''' + RQ'' = 0$ has general solution $Q(t) = c_1 + c_2 t + c_3 e^{-Rt/L}$. We had to solve $Lv' + Rv = 0$ as $v = \dfrac{c}{\text{integ. factor}} = c/e^{\frac{Rt}{L}}$.

$Q(0)=0$: $\quad c_1 + c_2(0) + c_3 e^0 = 0$

$Q'(0)=0$: $\quad 0c_1 + 1\cdot c_2 + \left(\frac{-R}{L}\right)c_3 e^0 = 0$

$Q''(0)=\frac{V_s}{L}$: $\quad 0c_1 + 0c_2 + \left(\frac{-R}{L}\right)^2 c_3 e^0 = \frac{V_s}{L}$

Then the matrix formulation is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -R/L \\ 0 & 0 & R^2/L^2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V_s/L \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -R/L & 0 \\ 0 & 0 & 1 & \frac{V_s L}{R^2} \end{array}\right) \quad \text{mult}(3, L^2/R^2)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & V_s/R \\ 0 & 0 & 1 & V_s L/R^2 \end{array}\right) \quad \text{combo}(3,2,R/L)$$

$$\text{combo}(3,1,-1) \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -V_s L/R^2 \\ 0 & 1 & 0 & V_s/R \\ 0 & 0 & 1 & V_s L/R^2 \end{array}\right)$$

$$\begin{cases} c_1 = -V_s L/R^2 \\ c_2 = V_s/R \\ c_3 = V_s L/R^2 \end{cases} \Rightarrow \boxed{Q(t) = \frac{-V_s L}{R^2} + \frac{V_s}{R}t + \frac{V_s L}{R^2}e^{-\frac{Rt}{L}}}$$

(c) Equil Sol is $I_\infty = V_s/R$. From above, $I = Q' = \frac{V_s}{R} - \frac{V_s}{R}e^{-\frac{Rt}{L}}$ which has limit V_s/R, also. They match.