Sample Quiz 5

Sample Quiz5 Problem 1. **Balancing Chemical Equations**: Tin oxide heated with hydrogen gas forms tin metal and water vapor.

The object of this problem is the derivation of the balanced chemical equation

$$\text{SnO2(s)} + 2\ \text{H2(g)} \rightarrow \text{Sn(s)} + 2\ \text{H2O(g)}.$$

The chemical symbols are defined as follows:

SnO2(s)	Tin dioxide	(s)=solid state	Ref: Tin dioxide
H2(g)	Hydrogen	(g)=gaseous state	Ref: Hydrogen
Sn(s)	Tin	(s)=solid state	Ref: Tin
H2O(g)	Water	(g)=gaseous state	Ref: Water vapor

Chemical balance means that matching elements on each side of a chemical equation have the same total amount of atoms. Notation 2 H2O documents 2x2=4 hydrogen atoms and 2x1=2 oxygen atoms. The given chemical equation is balanced, because the atom amounts for elements Sn, H, O total on each side of the equation to 1, 4, 2, respectively.

Chemical Balance by Linear Algebra. We begin with the unbalanced chemical equation

$$\text{SnO2(s)} + \text{H2(g)} \rightarrow \text{Sn(s)} + \text{H2O(g)},$$

which means tin dioxide and hydrogen gas chemically combine into tin and water vapor.

Let's write the balanced equation as

$$x_1\ \text{SnO2(s)} + x_2\ \text{H2(g)} \rightarrow x_3\ \text{Sn(s)} + x_4\ \text{H2O(g)},$$

where integers x_1 to x_4 are to be determined. Chemical balance means that the following algebraic equations hold:

$$\begin{array}{rcrl} x_1 & = & x_3 & \text{Tin Sn} \\ 2x_2 & = & 2x_4 & \text{Hydrogen H} \\ 2x_1 & = & x_4 & \text{Oxygen O} \end{array}$$

(a) Write the equations in standard matrix form $A\vec{x} = \vec{0}$, where A is a 3×4 matrix and $\vec{x}$ has components x_1 to x_4.

(b) Show details for computing the reduced row echelon form $\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}$

(c) There are more variables than equations. Therefore, there is at least one free variable. Identify the lead variables and the free variables.

(d) Solve the system $A\vec{x} = \vec{0}$. The answer is $x_1 = \frac{1}{2}t_1$, $x_2 = t_1$, $x_3 = \frac{1}{2}t_1$, $x_4 = t_1$, in terms of invented symbol t_1.

Balanced Equation. The smallest positive integer values for x_1 to x_4 are obtained from $t_1 = 2$. Then $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 2$. Substitution gives the balanced chemical equation

$$(1)\text{SnO2(s)} + (2)\text{H2(g)} \rightarrow (1)\text{Sn(s)} + (2)\text{H2O(g)}$$

References. Edwards-Penney section 5.1. Especially, this background theorem: Theorem 3: *If variable $\vec{x}$ has more entries than there are equations in system $A\vec{x} = \vec{0}$, then the system has infinitely many solutions.* Course document Linear Algebraic Equations, No Matrices. How to balance chemical equations, a video by Anne Marie Helmenstine with English transcript.

Sample Quiz5 Problem 2. **Solving Higher Order Initial Value Problems with Linear Algebra**.

The objective of this problem is to learn how to solve for the constants in a general solution, using given initial conditions. The differential equation and its known general solution are

$$y''' + 4y'' = 0,$$
$$y(x) = c_1 + c_2 x + c_3 e^{-4x}.$$

The initial conditions are

$$y(0) = 1, y'(0) = 2, y''(0) = -1.$$

(a) Derive the general solution formula, as follows. Make the substitution $v = y''$ to obtain $v' + 4v = 0$. Solve for v and then integrate twice to find y.

(b) Substitute the general solution into the three initial conditions to obtain the system of equations

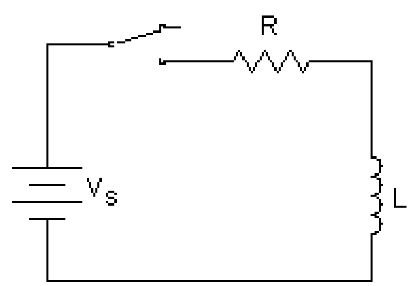
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 16 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(c) Solve the system of equations and then report the particular solution $y(x)$ so found.

> **Remark.** Picard's existence-uniqueness theorem says that the initial value problem has a unique solution, meaning that the linear algebra problem is in the unique solution case: no free variable.

References. Edwards-Penney sections 3.3, 3.4, 5.3. Course manuscript Linear Algebraic Equations, No Matrices.

Sample Quiz5 Problem 3. **An RL-Circuit with DC Voltage Source**.



The RL-circuit in the figure has model $LI' + RI = V_s$ where symbols L, R, V_s are constants. If the charge is $Q(t)$, then $I = Q'$ and $LQ'' + RQ' = V_s$. Differentiate this equation to obtain $LQ''' + RQ'' = 0$. Assume the current and charge are initially zero, before the switch is closed.

(a) Explain why Q has initial conditions $Q(0) = Q'(0) = 0$, $Q''(0) = V_s/L$.

(b) Solve for the charge $Q(t)$ and current $I(t)$ from the higher order equation $LQ''' + RQ'' = 0$ with initial conditions $Q(0) = Q'(0) = 0$, $Q''(0) = V_s/L$, guided by Problem 2, above. The system of linear equations to be solved is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{R}{L} \\ 0 & 0 & \frac{R^2}{L^2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{Vs}{L} \end{pmatrix}.$$

(c) The steady-state current $\lim_{t\to\infty} I(t)$ computed directly from your solution formula should equal the answer from the equilibrium solution of $LI' + RI = V_s$. Does it?

References. Edwards-Penney sections 3.7, 4.1, 5.1. Course manuscript Linear Algebraic Equations, No Matrices. Edwards-Penney BVP 3.7, electric circuit supplement. All About Circuits Volume I – DC by T. Kuphaldt. Course slides on Electric Circuits.