

Differential Equations 2280
Shortened Sample Final Exam
Wednesday, 6 May 2015, 12:45pm-3:15pm

Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

Chapters 1 and 2: Linear First Order Differential Equations

3. (Solve a Separable Equation)

Given $y^2 y' = \frac{2x^2 + 3x}{1 + x^2} \left(\frac{125}{64} - y^3 \right)$.

- (a) Find all equilibrium solutions.
(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly.

Answer:

(a) $y = 5/4$

(b)

$$-\frac{1}{3} \ln |125 - 64y^3| = 2x + \frac{3}{2} \ln(1 + x^2) - 2 \arctan(x) + c$$

4. (Linear Equations)

(a) [60%] Solve $2v'(t) = -32 + \frac{2}{3t+1}v(t)$, $v(0) = -8$. Show all integrating factor steps.

(b) [30%] Solve $2\sqrt{x+2} \frac{dy}{dx} = y$. The answer contains symbol c .

(c) [10%] The problem $2\sqrt{x+2} y' = y - 5$ can be solved using the answer y_h from part (b) plus superposition $y = y_h + y_p$. Find y_p .

Answer:

(a) $v(t) = -24t - 8$

(b) $y(x) = Ce^{\sqrt{x+2}}$

Chapter 3: Linear Equations of Higher Order

6. (ch3)

- (a) Solve for the general solutions:

(a.1) [25%] $y'' + 4y' + 4y = 0$,

(a.2) [25%] $y^{vi} + 4y^{iv} = 0$,

(a.3) [25%] Char. eq. $r(r - 3)(r^3 - 9r)^2(r^2 + 4)^3 = 0$.

(b) Given $6x''(t) + 7x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 6$, $c = 7$, $k = 2$, solve the differential equation [15%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a physical model drawing the meaning of constants m , c , k [5%].

Answer:

(a)

1: $r^2 + 4r + 4 = 0$, $y = c_1y_1 + c_2y_2$, $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.

2: $r^{iv} + 4r^2 = 0$, roots $r = 0, 0, 2i, -2i$. Then $y = c_1e^{0x} + c_2xe^{0x} + c_3 \cos 2x + c_4 \sin 2x$.

3: Write as $r^3(r - 3)^3(r + 3)^2(r^2 + 4)^3 = 0$. Then y is a linear combination of the atoms $1, x, x^2, e^{3x}, xe^{3x}, x^2e^{3x}, e^{-3x}, xe^{-3x}, \cos 2x, x \cos 2x, x^2 \cos 2x, \sin 2x, x \sin 2x, x^2 \sin 2x$.

Part (b)

Use $6r^2 + 7r + 2 = 0$ and the quadratic formula to obtain roots $r = -1/2, -2/3$. Then $x(t) = c_1e^{-t/2} + c_2e^{-2t/3}$. This is over-damped. The illustration shows a spring, dampener and mass with labels k, c, m, x and the equilibrium position of the mass.

7. (ch3)

Determine for $y^{vi} + y^{iv} = x + 2x^2 + x^3 + e^{-x} + x \sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Answer:

The homogeneous solution is a linear combination of the atoms $1, x, x^2, x^3, \cos x, \sin x$ because the characteristic polynomial has roots $0, 0, 0, 0, i, -i$.

1 An initial trial solution y is constructed by Rule I for atoms $1, x, e^{3x}, e^{-3x}, \cos x, \sin x$ giving

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 + d_2x + d_3x^2 + d_4x^3, \\ y_2 &= d_5 \cos x + d_6x \cos x, \\ y_3 &= d_7 \sin x + d_8x \sin x, \\ y_4 &= d_9e^{-x}. \end{aligned}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by specialization of constants, all derivatives of the right side of the differential equation.

2 Rule II is applied individually to each of y_1, y_2, y_3, y_4 .

The result is the **shortest trial solution**

$$\begin{aligned}y &= y_1 + y_2 + y_3 + y_4, \\y_1 &= d_1x^4 + d_2x^5 + d_3x^6 + d_4x^7, \\y_2 &= d_5x \cos x + d_6x^2 \cos x, \\y_3 &= d_7x \sin x + d_8x^2 \sin x, \\y_4 &= d_9e^{-x}.\end{aligned}$$

Chapters 4 and 5: Systems of Differential Equations

9. (ch5)

The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + c_3\mathbf{v}_3e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of A . Given

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix},$$

then

- (a) [75%] Display eigenanalysis details for A .
- (b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Answer:

(1): The eigenpairs are

$$\left(4, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right), \quad \left(6, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right), \quad \left(7, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right).$$

An expected detail is the cofactor expansion of $\det(A - \lambda I)$ and factoring to find eigenvalues 4, 6, 7. Eigenvectors should be found by a sequence of swap, combo, mult operations on the augmented matrix, followed by taking the partial ∂_{t_1} on invented symbol t_1 in the general solution to compute the eigenvector. In short, the eigenvectors are Strang's Special Solutions, and in general there can be many eigenvectors for a single eigenvalue.

(2): The eigenanalysis method for $\mathbf{x}' = A\mathbf{x}$ implies

$$\mathbf{x}(t) = c_1e^{4t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2e^{6t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3e^{7t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

10. (ch5)

(a) [20%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 4 & -2 \\ 0 & 0 & 2 \end{bmatrix}$.

(c) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton-Ziebur Method. In particular, display the equations that determine the three vectors in the general solution. **To save time**, don't solve for the three vectors in the formula. Only 2×2 on the final exam.

(d) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Eigenanalysis Method. **To save time**, find one eigenpair explicitly, just to show how it is done, but don't solve for the last two eigenpairs.

Answer:

(a) Eigenvalue Calculation

Subtract λ from the diagonal elements of A to obtain matrix $B = A - \lambda I$, then expand $\det(B)$ by cofactors to obtain the characteristic polynomial. The roots are the eigenvalues $\lambda = 2, 3, 5$.

(c) Cayley-Hamilton-Ziebur Method

The eigenvalues 2, 3, 5 from (a) are used to create the list of atoms e^{2t}, e^{3t}, e^{5t} . Then the Cayley-Hamilton-Ziebur method implies there are constant vectors $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \vec{\mathbf{d}}_3$ which depend on $\vec{\mathbf{u}}(0)$ and A such that

$$\vec{\mathbf{u}}(t) = e^{2t}\vec{\mathbf{d}}_1 + e^{3t}\vec{\mathbf{d}}_2 + e^{5t}\vec{\mathbf{d}}_3.$$

The determining equations are formed from differentiation of this formula two times, then replace $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, $\vec{\mathbf{u}}'' = A\vec{\mathbf{u}}' = AA\vec{\mathbf{u}}$. Finally, remove t from the three equations by setting $t = 0$, and define $\vec{\mathbf{u}}_0 = \vec{\mathbf{u}}(0)$. Then the three equations are, with $\mathbf{u}_0 = \mathbf{u}(0)$,

$$\begin{aligned} \vec{\mathbf{u}}_0 &= \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 + \vec{\mathbf{d}}_3 \\ A\vec{\mathbf{u}}_0 &= 2\vec{\mathbf{d}}_1 + 3\vec{\mathbf{d}}_2 + 5\vec{\mathbf{d}}_3 \\ A^2\vec{\mathbf{u}}_0 &= 4\vec{\mathbf{d}}_1 + 9\vec{\mathbf{d}}_2 + 25\vec{\mathbf{d}}_3 \end{aligned}$$

This ends the solution to the problem. We continue, solving for the vectors $\vec{\mathbf{d}}_j$, just to illustrate how it is done. The matrix of coefficients

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{pmatrix}$$

and its transpose matrix $B = C^T$ give a formal relation

$$\mathbf{aug}(\vec{\mathbf{u}}_0, A\vec{\mathbf{u}}_0, A^2\vec{\mathbf{u}}_0) = \mathbf{aug}(\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \vec{\mathbf{d}}_3)B.$$

Multiplying this relation by B^{-1} gives

$$\mathbf{aug}(\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \vec{\mathbf{d}}_3) = \mathbf{aug}(\vec{\mathbf{u}}_0, A\vec{\mathbf{u}}_0, A^2\vec{\mathbf{u}}_0)B^{-1}.$$

Then disassembling the formal matrix multiply implies

$$\begin{aligned}\vec{\mathbf{d}}_1 &= 5\vec{\mathbf{u}}_0 - \frac{8}{3}A\vec{\mathbf{u}}_0 + \frac{1}{3}A^2\vec{\mathbf{u}}_0 \\ \vec{\mathbf{d}}_2 &= -5\vec{\mathbf{u}}_0 + \frac{7}{2}A\vec{\mathbf{u}}_0 - \frac{1}{2}A^2\vec{\mathbf{u}}_0 \\ \vec{\mathbf{d}}_3 &= 5\vec{\mathbf{u}}_0 - \frac{5}{6}A\vec{\mathbf{u}}_0 + \frac{1}{6}A^2\vec{\mathbf{u}}_0\end{aligned}$$

The matrix of coefficients is

$$\begin{pmatrix} 5 & -\frac{8}{3} & \frac{1}{3} \\ -5 & \frac{7}{2} & -\frac{1}{2} \\ 1 & -\frac{5}{6} & \frac{1}{6} \end{pmatrix} = (B^{-1})^T = C^{-1}!$$

This fact, that solving for $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \vec{\mathbf{d}}_3$ in the displayed equations reduces to inverting the matrix of coefficients, can be used as a shortcut in the Cayley-Hamilton-Ziebur method.

(d) Eigenanalysis Method

For matrix

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

the eigenpairs are computed to be

$$\left(2, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right), \left(3, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right), \left(5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right).$$

Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ has general solution

$$\vec{\mathbf{u}}(t) = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

(e) Laplace's Method

The start is the Laplace resolvent formula for matrix differential equation $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$.

$$(sI - A) \mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}_0.$$

This formula expands to

$$\begin{pmatrix} s-4 & -1 & 1 \\ -1 & s-4 & 2 \\ 0 & 0 & s-2 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \\ \mathcal{L}(z) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where symbols a, b, c are arbitrary constants for the initial data $\vec{\mathbf{u}}_0$. Let W denote the coefficient matrix. Then the inverse of W can be computed using the adjugate formula $W^{-1} = \mathbf{adj}(W)/\det(W)$. The answer for the inverse is

$$W^{-1} = \frac{1}{(s-5)(s-2)(s-3)} \begin{pmatrix} s^2 - 6s + 8 & s-2 & -s+2 \\ s-2 & s^2 - 6s + 8 & -2s+7 \\ 0 & 0 & s^2 - 8s + 15 \end{pmatrix}$$

True, this formula can be derived and then followed by inverse Laplace methods to obtain an answer in variable t . However, we already know the outcome, because this matrix is the Laplace of the exponential matrix e^{At} . The exponential matrix formula was already derived in (b) above. Expanding the matrix multiplies and collecting terms gives the final answer

$$W^{-1} = \mathcal{L}(e^{At}) = \frac{1}{2} \mathcal{L} \begin{pmatrix} e^{5t} + e^{3t} & e^{5t} - e^{3t} & -e^{5t} + e^{3t} \\ e^{5t} - e^{3t} & e^{5t} + e^{3t} & -e^{5t} + 2e^{2t} - e^{3t} \\ 0 & 0 & 2e^{2t} \end{pmatrix}$$

Canceling the \mathcal{L} with Lerch's Theorem implies the same answer as found in part (b), which is

$$\vec{\mathbf{u}}(t) = e^{At} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{5t} + e^{3t} & e^{5t} - e^{3t} & -e^{5t} + e^{3t} \\ e^{5t} - e^{3t} & e^{5t} + e^{3t} & -e^{5t} + 2e^{2t} - e^{3t} \\ 0 & 0 & 2e^{2t} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

11. (ch5)

(a) [50%] The eigenvalues are 4, 6 for the matrix $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$. Show details from either the eigenanalysis method or the Laplace method.

(b) [50%] Using the same matrix A from part (a), display the solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton Method. To save time, write out the system to be solved for the two vectors, and then stop, without solving for the vectors.

(c) [50%] Using the same matrix A from part (a), compute the exponential matrix e^{At} by any known method, for example, the formula $e^{At} = \Phi(t)\Phi^{-1}(0)$ where $\Phi(t)$ is any fundamental matrix, or via Putzer's formula.

Answer:

(a) Eigenanalysis method

The eigenpairs of A are

$$\left(4, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right), \quad \left(6, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

which implies the eigenanalysis general solution

$$\mathbf{u}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) Cayley-Hamilton-Ziebur method

Then $\mathbf{u}(t) = e^{4t}\vec{c}_1 + e^{6t}\vec{c}_2$ for some constant vectors \vec{c}_1, \vec{c}_2 that depend on $\vec{u}(0)$ and A . Differentiate this equation once and use $\vec{u}' = A\vec{u}$, then set $t = 0$. The resulting system is

$$\begin{aligned}\vec{u}_0 &= e^0\vec{c}_1 + e^0\vec{c}_2 \\ A\vec{u}_0 &= 4e^0\vec{c}_1 + 6e^0\vec{c}_2\end{aligned}$$

(c) Putzer Method

The result is $e^{At} = e^{4t}I + \frac{e^{4t}-e^{6t}}{4-6}(A-4I)$. Functions r_1, r_2 are computed from $r_1' = 4r_1$, $r_1(0) = 1$, $r_2' = 6r_2 + r_1$, $r_2(0) = 0$.

$$e^{At} = \frac{1}{2} \begin{pmatrix} e^{4t} + e^{6t} & e^{6t} - e^{4t} \\ e^{6t} - e^{4t} & e^{4t} + e^{6t} \end{pmatrix}.$$

12. (ch5)

(a) [50%] Display the solution of $\mathbf{u}' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, using any method that applies.

Answer:

(a) **Resolvent method**

The resolvent equation $(sI - A)\mathcal{L}(\vec{u}) = \vec{u}(0)$ is the system

$$\begin{pmatrix} s-2 & 0 \\ -1 & s-2 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The system is solved by Cramer's rule for unknowns $\mathcal{L}(x), \mathcal{L}(y)$ to obtain

$$\mathcal{L}(x) = \frac{0}{(s-2)^2}, \quad \mathcal{L}(y) = \frac{s-2}{(s-2)^2}.$$

The backward Laplace table implies

$$x(t) = 0, \quad y(t) = e^{2t}.$$

Best method. Look at the equations as scalar equations $x' = 2x$, $x(0) = 0$ and $y' = x + 2y$, $y(0) = 1$. Clearly $x(t) = 0$ and then $y' = 0 + 2y$, $y(0) = 1$ implies $y(t) = e^{2t}$.

Chapter 6: Dynamical Systems

14. (ch6) Only half of these items appear on the final exam.

Find the equilibrium points of $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$ and classify

each linearization at an equilibrium as a node, spiral, center, saddle. What classifications can be deduced for the nonlinear system, according to the Poincaré Theorem?

Answer:

The equilibria are constant solutions, which are found from the equations

$$\begin{aligned} 0 &= (14 - x/2 - y)x \\ 0 &= (16 - y/2 - x)y \end{aligned}$$

Considering when a zero factor can occur leads to the four equilibria $(0, 0)$, $(0, 32)$, $(28, 0)$, $(12, 8)$. The last equilibrium comes from solving the system of equations

$$\begin{aligned} x/2 + y &= 14 \\ x + y/2 &= 16 \end{aligned}$$

Linearization

The Jacobian matrix J is the augmented matrix of partial derivatives $\partial_x \vec{F}$, $\partial_y \vec{F}$ (column vectors) computed from

$$\vec{f}(x, y) = \begin{pmatrix} 14x - x^2/2 - yx \\ 16y - y^2/2 - xy \end{pmatrix}.$$

Then

$$J(x, y) = \begin{pmatrix} 14 - x - y & -x \\ -y & 16 - y - x \end{pmatrix}.$$

The four matrices below are $J(x, y)$ when (x, y) is replaced by an equilibrium point. Included in the table are the roots of the characteristic equation for each matrix and its classification based on the roots. No book was consulted for the classifications. The idea in each is to examine the limits at $t = \pm\infty$, then eliminate classifications. No matrix has complex eigenvalues, and that eliminates the center and spiral. The first three are stable at either $t = \infty$ or $t = -\infty$, which eliminates the saddle and leaves the node as the only possible classification.

$$\begin{aligned} A_1 &= J(0, 0) &= \begin{pmatrix} 14 & 0 \\ 0 & 16 \end{pmatrix} & r = 14, 16 & \text{node} \\ A_2 &= J(0, 32) &= \begin{pmatrix} -18 & 0 \\ -32 & -16 \end{pmatrix} & r = -18, -16 & \text{node} \\ A_3 &= J(28, 0) &= \begin{pmatrix} -14 & -28 \\ 0 & -12 \end{pmatrix} & r = -14, -12 & \text{node} \\ A_4 &= J(12, 8) &= \begin{pmatrix} -6 & -12 \\ -8 & -4 \end{pmatrix} & r = -5 + \sqrt{97}, -5 - \sqrt{97} & \text{saddle} \end{aligned}$$

Some maple code for checking the answers:


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F:=unapply([14*x-x^2/2-y*x , 16*y-y^2/2 -x*y],(x,y));
Fx:=unapply(map(u->diff(u,x),F(x,y)),(x,y));
Fy:=unapply(map(u->diff(u,y),F(x,y)),(x,y));
Fx(0,0);Fy(0,0);Fx(28,0);Fy(28,0);Fx(0,32);Fy(0,32);Fx(0,32);Fy(0,32);

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15. (ch6) Only half of these items appear on the final exam.

(a) [25%] Which of the four types *center*, *spiral*, *node*, *saddle* can be unstable at $t = \infty$? Explain your answer.

(b) [25%] Give an example of a linear 2-dimensional system $\mathbf{u}' = A\mathbf{u}$ with a saddle at equilibrium point $x = y = 0$, and A is not triangular.

(c) [25%] Give an example of a nonlinear 2-dimensional predator-prey system with exactly four equilibria.

(d) [25%] Display a formula for the general solution of the equation $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{u}$.

Then explain why the system has a spiral at $(0, 0)$.

(e) [25%] Is the origin an isolated equilibrium point of the linear system $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{u}$? Explain your answer.

Answer:

(a) All except the center, which is stable but not asymptotically stable. All the others correspond to a general solution which can have an exponential factor e^{kt} in each term. If $k > 0$, then the solution cannot approach the origin at $t = \infty$.

(b) Required are characteristic roots like 1, -1. Let $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Define $A = PBP^{-1}$ where $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\mathbf{u}' = A\mathbf{u}$ has a saddle at the origin, because the

characteristic roots of A are still 1, -1. And $A = \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$ is not triangular.

(c) **Example:** The nonlinear predator-prey system $x' = (x+y-4)x$, $y' = (-x+2y-2)y$ has exactly four equilibrium points $(0, 0)$, $(4, 0)$, $(0, 1)$, $(2, 2)$.

(d) The characteristic equation $\det(A - \lambda I) = 0$ is $(1 - \lambda)^2 + 1 = 0$ with complex roots $1 \pm i$ and corresponding atoms $e^t \cos t$, $e^t \sin t$. Then the Cayley-Hamilton-Ziebur Method implies

$$\vec{\mathbf{u}}(t) = e^t \cos t \vec{\mathbf{c}}_1 + e^t \sin t \vec{\mathbf{c}}_2.$$

Explanation, why the classification is a spiral. Such solutions containing sine and cosine factors wrap around the origin. This makes it a spiral or a center. Because of the exponential factor e^t , it is asymptotically stable at $t = -\infty$, which disallows a center, so it is a spiral.

(e) No, because $\det(A) = 0$. In this case, $A\mathbf{u} = \mathbf{0}$ has infinitely many solutions, describing a line of equilibria through the origin. This implies the equilibrium point $(0, 0)$ is not isolated [you cannot draw a circle about $(0, 0)$ which contains no other equilibrium point].

Chapter 7: Laplace Theory

16. (ch7)

(d) Explain all the steps in Laplace's Method, as applied to the differential equation $x'(t) + 2x(t) = e^t$, $x(0) = 1$.

Answer:

(d) Laplace's method explained.

The first step transforms the equation using the parts formula and initial data to get

$$(s + 2)\mathcal{L}(x) = 1 + \mathcal{L}(e^t).$$

The forward Laplace table applies to write, after a division, the isolated formula for $\mathcal{L}(x)$:

$$\mathcal{L}(x) = \frac{1 + 1/(s - 1)}{s + 2} = \frac{s}{(s - 1)(s + 2)}.$$

Partial fraction methods imply

$$\mathcal{L}(x) = \frac{a}{s - 1} + \frac{b}{s + 2} = \mathcal{L}(ae^t + be^{-2t})$$

and then $x(t) = ae^t + be^{-2t}$ by Lerch's theorem. The constants are $a = 1/3$, $b = 2/3$.

17. (ch7) Only half of the items appear on the final exam.

(a) Solve $\mathcal{L}(f(t)) = \frac{100}{(s^2 + 1)(s^2 + 4)}$ for $f(t)$.

(b) Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \frac{1}{s^2(s - 3)}$.

(c) Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t} \sin(3t)$.

(d) Find $\mathcal{L}(f)$ where $f(t)$ is the periodic function of period 2 equal to $t/2$ on $0 \leq t \leq 2$ (sawtooth wave).

Answer:

(a) $\mathcal{L}(f) = \frac{100}{(u+1)(u+4)} = \frac{100/3}{u+1} + \frac{-100/3}{u+4}$ where $u = s^2$. Then $\mathcal{L}(f) = \frac{100}{3} \left(\frac{1}{s^2+1} - \frac{1}{s^2+4} \right) = \frac{100}{3} \mathcal{L}(\sin t - \frac{1}{2} \sin 2t)$ implies $f(t) = \frac{100}{3} (\sin t - \frac{1}{2} \sin 2t)$.

(b) $\mathcal{L}(f) = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-3} = \mathcal{L}(a + bt + ce^{3t})$ implies $f(t) = a + bt + ce^{3t}$. The constants, by Heaviside coverup, are $a = -1/9$, $b = -1/3$, $c = 1/9$.

(c) $\mathcal{L}(f) = \frac{d}{ds} \mathcal{L}(e^{2t} \sin 3t)$ by the s -differentiation theorem. The first shifting theorem implies $\mathcal{L}(e^{2t} \sin 3t) = \mathcal{L}(\sin 3t)|_{s \rightarrow (s-2)}$. Finally, the forward table implies $\mathcal{L}(f) = \frac{d}{ds} \left(\frac{1}{(s-2)^2+9} \right) = \frac{-2(s-2)}{((s-2)^2+9)^2}$.

18. (ch7)

(a) Solve $y'' + 4y' + 4y = t^2$, $y(0) = y'(0) = 0$ by Laplace's Method.

(c) Solve the system $x' = x + y$, $y' = x - y + e^t$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

Answer:

(a) Transform to get $\mathcal{L}(x) = \frac{\mathcal{L}(t^2)}{s^2+4s+4}$. Then $\mathcal{L}(x) = \frac{1}{s^3(s+2)^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s^3} + \frac{d}{s+2} + \frac{f}{(s+2)^2} = \mathcal{L}(a + bt + ct^2 + de^{-2t} + fte^{-2t})$. The answer is $x(t) = a + bt + ct^2 + de^{-2t} + fte^{-2t}$.

The partial fraction constants are $a = 3/16$, $b = -1/4$, $c = 1/4$, $d = -3/16$, $f = -1/8$.

(c) Transform to get $\mathcal{L}(x) = \frac{1}{s^3+s^2-6s} = \frac{1}{s(s-2)(s+3)} = \frac{a}{s} + \frac{b}{s-2} + \frac{c}{s+3} = \mathcal{L}(a+be^{2t}+ce^{-3t})$.

Then the answer is $x(t) = a + be^{2t} + ce^{-3t}$. The partial fraction constants are $a = -1/6$, $b = 1/10$, $c = 1/15$.

19. (ch7)

(a) [50%] Solve by Laplace's method $x'' + x = \cos t$, $x(0) = x'(0) = 0$.

(d) [50%] Solve by Laplace's resolvent method

$$\begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 2x(t), \end{aligned}$$

with initial conditions $x(0) = -1$, $y(0) = 2$.

Answer:

(a) Transform to obtain $\mathcal{L}(x) = \frac{s}{(s^2+1)^2}$.

Calculus method. Observe that $\frac{d}{ds} \frac{1}{s^2+1} = \frac{-2s}{(s^2+1)^2}$. Then $\mathcal{L}(x) = -\frac{1}{2} \frac{d}{ds} \frac{1}{s^2+1} = -\frac{1}{2} \frac{d}{ds} \mathcal{L}(\sin t) = -\frac{1}{2} \mathcal{L}((-t) \sin t)$ by the s -differentiation theorem. Finally, $x(t) = \frac{1}{2} t \sin t$.

Convolution method. Write $\mathcal{L}(x) = \mathcal{L}(\sin t) \mathcal{L}(\cos t)$. Apply the convolution theorem to obtain $x(t) = \int_0^t \sin u \cos(t-u) du = \frac{1}{2} t \sin t$. A maple answer check is

$$\text{int}(\sin(u)*\cos(t-u), u=0..t);$$

Hand integration uses the trigonometric identity $2 \sin(a) \cos(b) = \cos(a-b) - \cos(a+b)$.

(d) The resolvent formula $(sI - A)\mathcal{L}(\vec{u}) = \vec{u}_0$ becomes the $2 \times$ system of equations

$$\begin{pmatrix} s-1 & -1 \\ -2 & s-0 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Multiply by the inverse matrix of $(sI - A)$ on the left to obtain

$$\begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} s-1 & -1 \\ -2 & s-0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} s-0 & 1 \\ 2 & s-1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix},$$

where $\Delta = \det(sI - A) = (s+1)(s-2)$. Then $\mathcal{L}(x) = \frac{2-s}{\Delta} = \frac{-1}{s+1}$, $\mathcal{L}(y) = \frac{2s}{\Delta} = \frac{2s-4}{(s+1)(s-2)} = \frac{2}{s+1}$. Then $x(t) = -e^{-t}$, $y(t) = 2e^{-t}$.

20. (ch7) Fewer items appear on the final exam.

(a) [25%] Solve $\mathcal{L}(f(t)) = \frac{10}{(s^2 + 8)(s^2 + 4)}$ for $f(t)$.

(b) [25%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \frac{s + 1}{s^2(s + 2)}$.

(c) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \frac{s - 1}{s^2 + 2s + 5}$.

(d) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \frac{d}{ds} \mathcal{L}(t^2 \sin 3t)$$

(e) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \left(\mathcal{L}(t^3 e^{9t} \cos 8t) \right) \Big|_{s \rightarrow s+3}.$$

Answer:

(a) $\mathcal{L}(f(t)) = \frac{10}{u+8}u + 4$ where $u = s^2$. Use Heaviside's coverup method to find the partial fraction expansion

$$\frac{10}{u+8}u + 4 = \frac{-5/2}{u+8} + \frac{5/2}{u+4} = \frac{-5/2}{s^2+8} + \frac{5/2}{s^2+4}.$$

Then $\mathcal{L}(f(t)) = \mathcal{L}\left(-\frac{5}{2} \frac{\sin \sqrt{8}t}{\sqrt{8}} + \frac{5}{2} \frac{\sin 2t}{2}\right)$ implies by Lerch's theorem

$$f(t) = -\frac{5}{2} \frac{\sin \sqrt{8}t}{\sqrt{8}} + \frac{5}{2} \frac{\sin 2t}{2}.$$

(b) Expand the fraction into partial fractions as follows:

$$\mathcal{L}(f) = \frac{s+1}{s^2(s+2)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+2} = \mathcal{L}(a + bt + ce^{-2t}).$$

Then Lerch's theorem implies $f(t) = a + bt + ce^{-2t}$. The partial fraction constants are $a = 1/4, b = 1/2, c = -1/4$.

(d) Because $\frac{d}{ds} \mathcal{L}(g(t)) = \mathcal{L}((-t)g(t))$, then $\mathcal{L}(f) = \mathcal{L}((-t)t^2 \sin 3t)$. Lerch's theorem implies $f(t) = -t^3 \sin 3t$.

(e) The shifting theorem $\mathcal{L}(g(t))|_{s \rightarrow (s-a)} = \mathcal{L}(e^{at}g(t))$ is applied to remove the shift on the outside and put e^{-3t} into the Laplace integrand. Then $\mathcal{L}(f(t)) = \mathcal{L}(e^{-3t}t^3 e^{9t} \cos 8t)$. Lerch's theorem implies $f(t) = t^3 e^{6t} \cos 8t$.

Chapter 9: Fourier Series and Partial Differential Equations

21. (ch9)

(b) State Fourier's convergence theorem.

(c) State the results for term-by-term integration and differentiation of Fourier series.

Answer:

(a) Let $f(x)$ be piecewise smooth and 2π -periodic on the whole real line. Then

$$\frac{1}{2} (f(x+) + f(x-)) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

where the Fourier coefficients are given by the inner product formulas

$$a_n = \frac{\langle f(x), \cos(nx) \rangle}{\langle \cos(nx), \cos(nx) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad b_n = \frac{\langle f(x), \sin(nx) \rangle}{\langle \sin(nx), \sin(nx) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

The convergence is pointwise on the whole real line.

(b) INTEGRATION. Assume $f(x)$ is piecewise continuous and periodic of period $2L$ on the whole real line. Then

$$\int_0^t f(x) dx = \frac{a_0}{2} t + \frac{a_0}{2} + \sum_{n=1}^{\infty} \int_0^t (a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)) dt$$

where the Fourier coefficients are given by the inner product formulas

$$a_n = \frac{\langle f(x), \cos(n\pi x/L) \rangle}{\langle \cos(n\pi x/L), \cos(n\pi x/L) \rangle} = \frac{1}{L} \int_{-L}^L f(x) \cos(n\pi x/L) dx, \quad b_n = \frac{\langle f(x), \sin(n\pi x/L) \rangle}{\langle \sin(n\pi x/L), \sin(n\pi x/L) \rangle} = \frac{1}{L} \int_{-L}^L f(x) \sin(n\pi x/L) dx$$

(b) DIFFERENTIATION.

22. (ch9)

(c) Solve $u_t = u_{xx}$, $u(0, t) = u(\pi, t) = 0$, $u(x, 0) = 80 \sin^3 x$ on $0 \leq x \leq \pi$, $t \geq 0$.

Answer:

(c)

A possible shortcut trick is to apply Fourier's exponential insertion method, after writing $f(x)$ as a sine series. This uses trig identities, as follows.

$$\begin{aligned} \sin^3(x) &= (1 - \cos^2(x)) \sin(x) = \sin(x) - \frac{1}{2} \cos(x)(2 \sin(x) \cos(x)) \\ &= \sin(x) - \frac{1}{2} \cos(x) \sin(2x) \\ &= \sin(x) - \frac{1}{2} \frac{1}{2} (\sin(2x + x) + \sin(2x - x)) \\ &= \sin(x) - \frac{1}{4} \sin(3x) + \frac{1}{4} \sin(x) \\ &= \frac{3}{4} \sin(x) - \frac{1}{4} \sin(3x) \end{aligned}$$

Then

$$f(x) = 80 \left(\frac{3}{4} \sin(x) - \frac{1}{4} \sin(3x) \right) = 60 \sin(x) - 20 \sin(3x).$$

Fourier's method inserts the correct exponential factors to obtain the answer

$$u(x, t) = 60 \sin(x) e^{-t} - 20 \sin(3x) e^{-9t}.$$

23. (Vibration of a Finite String)

The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x, t)$ equal to an infinite series of constants times normal modes.

Solve the finite string vibration problem on $0 \leq x \leq 2$, $t > 0$,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0, t) &= 0, \\ u(2, t) &= 0, \\ u(x, 0) &= 0, \\ u_t(x, 0) &= -11 \sin(5\pi x). \end{aligned}$$

Answer: Because the wave initial shape is zero, then the only normal modes are sine times sine. The initial wave velocity is already a Fourier series, using orthogonal set $\{\sin(n\pi x/2)\}_{n=1}^{\infty}$. The 1-term Fourier series $-11 \sin(5\pi x)$ can be modified into a solution by inserting the missing sine factor present in the corresponding normal mode. Then $u(x, t) = -11 \sin(5\pi x) \sin(5\pi ct)/(5\pi)$. We check it is a solution.

24. (Periodic Functions)

(c) [30%] Mark the expressions which are periodic with letter **P**, those odd with **O** and those even with **E**.

$$\sin(\cos(2x)) \quad \ln|2 + \sin(x)| \quad \sin(2x) \cos(x) \quad \frac{1 + \sin(x)}{2 + \cos(x)}$$

Answer:

(c) All are periodic of period 2π , satisfying $f(x + 2\pi) = f(x)$. The first is even and the third is odd.

25. (Fourier Series)

Let $f_0(x) = x$ on the interval $0 < x < 2$, $f_0(x) = -x$ on $-2 < x < 0$, $f_0(x) = 0$ for $x = 0$, $f_0(x) = 2$ at $x = \pm 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(a) [80%] Compute the Fourier coefficients of $f(x)$ (defined above) for the terms $\sin(67\pi x)$ and $\cos(2\pi x)$. Leave tedious integrations in integral form, but evaluate the easy ones like the integral of the square of sine or cosine.

(b) [20%] Which values of x in $|x| < 12$ might exhibit Gibb's over-shoot?

Answer: (a) Because $f_0(x)$ is even, then $f(x)$ is even. Then the coefficient of $\sin(67\pi x)$ is zero, without computation, because all sine terms in the Fourier series of f have zero coefficient. The coefficient of $\cos(n\pi x/2)$ for $n > 0$ is given by the formula

$$a_n = \frac{1}{2} \int_{-2}^2 f_0(x) \cos(n\pi x/2) dx = \int_0^2 x \cos(n\pi x/2) dx.$$

For $\cos(2\pi x)$, we select $n\pi x/2 = 2\pi x$, or index $n = 4$.

(b) There are no jump discontinuities, f is continuous, so no Gibbs overshoot.

27. (Convergence of Fourier Series)

(c) [30%] Give an example of a function $f(x)$ periodic of period 2 that has a Gibb's over-shoot at the integers $x = 0, \pm 2, \pm 4, \dots$, (all $\pm 2n$) and nowhere else.

Answer:

(c) Any 2-periodic continuous function f will work, if we alter the values of f at the desired points to produce a jump discontinuity. For example, define $f(x) = \sin(\pi x)$ except at the points $\pm 2n$, where $f(2n) = 2$ ($n = 0, 1, 2, 3, \dots$).
