Differential Equations 2280
Shortened Sample Final Exam
Wednesday, 6 May 2015, 12:45pm-3:15pm

Instructions: This in-class exam is 120 minutes. About 20 minutes per sub-section. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

Chapters 1 and 2: Linear First Order Differential Equations

3. (Solve a Separable Equation)
Given $y^2y' = \dfrac{2x^2+3x}{1+x^2}\left(\dfrac{125}{64} - y^3\right)$.
(a) Find all equilibrium solutions.
(b) Find the non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly.

4. (Linear Equations)
(a) [60%] Solve $2v'(t) = -32 + \dfrac{2}{3t+1}v(t)$, $v(0) = -8$. Show all integrating factor steps.
(b) [30%] Solve $2\sqrt{x+2}\,\dfrac{dy}{dx} = y$. The answer contains symbol c.
(c) [10%] The problem $2\sqrt{x+2}\,y' = y - 5$ can be solved using the answer y_h from part (b) plus superposition $y = y_h + y_p$. Find y_p.

Chapter 3: Linear Equations of Higher Order

6. (ch3)
(a) Solve for the general solutions:

(**a.1**) [25%] $y'' + 4y' + 4y = 0$,

(**a.2**) [25%] $y^{vi} + 4y^{iv} = 0$,

(**a.3**) [25%] Char. eq. $r(r-3)(r^3-9r)^2(r^2+4)^3 = 0$.

(b) Given $6x''(t) + 7x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 6$, $c = 7$, $k = 2$, solve the differential equation [15%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a physical model drawing the meaning of constants m, c, k [5%].

7. (ch3)
Determine for $y^{vi} + y^{iv} = x + 2x^2 + x^3 + e^{-x} + x\sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Chapters 4 and 5: Systems of Differential Equations

9. (ch5)
The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix},$$

then
(a) [75%] Display eigenanalysis details for A.
(b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

10. (ch5)

(a) [20%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 4 & -2 \\ 0 & 0 & 2 \end{bmatrix}$.

(c) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton-Ziebur Method. In particular, display the equations that determine the three vectors in the general solution. **To save time**, don't solve for the three vectors in the formula. Only 2×2 on the final exam.

(d) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Eigenanalysis Method. **To save time**, find one eigenpair explicitly, just to show how it is done, but don't solve for the last two eigenpairs.

11. (ch5)
(a) [50%] The eigenvalues are 4, 6 for the matrix $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$. Show details from either the eigenanalysis method or the Laplace method.

(b) [50%] Using the same matrix A from part (a), display the solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton Method. To save time, write out the system to be solved for the two vectors, and then stop, without solving for the vectors.

(c) [50%] Using the same matrix A from part (a), compute the exponential matrix e^{At} by any known method, for example, the formula $e^{At} = \Phi(t)\Phi^{-1}(0)$ where $\Phi(t)$ is any fundamental matrix, or via Putzer's formula.

12. (ch5)
(a) [50%] Display the solution of $\mathbf{u}' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, using any method that applies.

Chapter 6: Dynamical Systems

14. (ch6) Only half of these items appear on the final exam.
Find the equilibrium points of $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$ and classify each linearization at an equilibrium as a node, spiral, center, saddle. What classifications can be deduced for the nonlinear system, according to the Paste Theorem?

Some maple code for checking the answers:

```
F:=unapply([14*x-x^2/2-y*x , 16*y-y^2/2 -x*y],(x,y));
Fx:=unapply(map(u->diff(u,x),F(x,y)),(x,y));
Fy:=unapply(map(u->diff(u,y),F(x,y)),(x,y));
Fx(0,0);Fy(0,0);Fx(28,0);Fy(28,0);Fx(0,32);Fy(0,32);Fx(0,32);Fy(0,32);
```

15. (ch6) Only half of these items appear on the final exam.
(a) [25%] Which of the four types *center, spiral, node, saddle* can be unstable at $t = \infty$? Explain your answer.
(b) [25%] Give an example of a linear 2-dimensional system $\mathbf{u}' = A\mathbf{u}$ with a saddle at equilibrium point $x = y = 0$, and A is not triangular.
(c) [25%] Give an example of a nonlinear 2-dimensional predator-prey system with exactly four equilibria.
(d) [25%] Display a formula for the general solution of the equation $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{u}$.
Then explain why the system has a spiral at $(0, 0)$.
(e) [25%] Is the origin an isolated equilibrium point of the linear system $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{u}$? Explain your answer.

Chapter 7: Laplace Theory

16. (ch7)

(d) Explain all the steps in Laplace's Method, as applied to the differential equation $x'(t) + 2x(t) = e^t$, $x(0) = 1$.

17. (ch7) Only half of the items appear on the final exam.

(a) Solve $\mathcal{L}(f(t)) = \dfrac{100}{(s^2+1)(s^2+4)}$ for $f(t)$.

(b) Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s-3)}$.

(c) Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}\sin(3t)$.

(d) Find $\mathcal{L}(f)$ where $f(t)$ is the periodic function of period 2 equal to $t/2$ on $0 \le t \le 2$ (sawtooth wave).

18. (ch7)

(a) Solve $y'' + 4y' + 4y = t^2$, $y(0) = y'(0) = 0$ by Laplace's Method.

(c) Solve the system $x' = x + y$, $y' = x - y + e^t$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

19. (ch7)
(a) [50%] Solve by Laplace's method $x'' + x = \cos t$, $x(0) = x'(0) = 0$.

(d) [50%] Solve by Laplace's resolvent method

$$\begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 2x(t), \end{aligned}$$

with initial conditions $x(0) = -1$, $y(0) = 2$.

20. (ch7) Fewer items appear on the final exam.
(a) [25%] Solve $\mathcal{L}(f(t)) = \dfrac{10}{(s^2+8)(s^2+4)}$ for $f(t)$.

(b) [25%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2(s+2)}$.

(c) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s-1}{s^2+2s+5}$.

(d) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(t^2 \sin 3t)$$

(e) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \left(\mathcal{L}\left(t^3 e^{9t} \cos 8t\right)\right)\Big|_{s \to s+3}.$$

Chapter 9: Fourier Series and Partial Differential Equations

21. (ch9)
(b) State Fourier's convergence theorem.
(c) State the results for term-by-term integration and differentiation of Fourier series.

22. (ch9)
(c) Solve $u_t = u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = 80\sin^3 x$ on $0 \le x \le \pi$, $t \ge 0$.

23. (Vibration of a Finite String)
The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes.

Solve the finite string vibration problem on $0 \le x \le 2$, $t > 0$,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0,t) &= 0, \\ u(2,t) &= 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= -11\sin(5\pi x). \end{aligned}$$

24. (Periodic Functions)
(c) [30%] Mark the expressions which are periodic with letter **P**, those odd with **O** and those even with **E**.

$$\sin(\cos(2x)) \qquad \ln|2+\sin(x)| \qquad \sin(2x)\cos(x) \qquad \frac{1+\sin(x)}{2+\cos(x)}$$

25. (Fourier Series)
Let $f_0(x) = x$ on the interval $0 < x < 2$, $f_0(x) = -x$ on $-2 < x < 0$, $f_0(x) = 0$ for $x = 0$, $f_0(x) = 2$ at $x = \pm 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(a) [80%] Compute the Fourier coefficients of $f(x)$ (defined above) for the terms $\sin(67\pi x)$ and $\cos(2\pi x)$. Leave tedious integrations in integral form, but evaluate the easy ones like the integral of the square of sine or cosine.

(b) [20%] Which values of x in $|x| < 12$ might exhibit Gibb's over-shoot?

27. (Convergence of Fourier Series)
(c) [30%] Give an example of a function $f(x)$ periodic of period 2 that has a Gibb's over-shoot at the integers $x = 0, \pm 2, \pm 4, \ldots$, (all $\pm 2n$) and nowhere else.