Differential Equations 2280

Midterm Exam 3

Exam Date: 24 April 2015 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Chapter 3

1. **(Linear Constant Equations of Order n)**

(a) [30%] Find by variation of parameters a particular solution y_p for the equation $y'' = 2 + 6x$. Show all steps in variation of parameters. Check the answer by quadrature.

(b) [10%] A particular solution of the equation $LI'' + RI' + (1/C)I = I_0 \cos(10t)$ happens to be $I(t) = 5\cos(10t) + e^{-2t}\sin(\sqrt{17}t) - \sqrt{17}\sin(10t)$. Assume L, R, C all positive. Find the unique periodic steady-state solution $I_{\rm SS}$.

(c) [40%] Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 64x = 39\cos(5t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time**, please don't convert to phase-amplitude form.

(d) [10%] Given $5x''(t) + 2x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 5$, $c = 2$, $k = 2$, determine if the equation is over-damped , critically damped or under-damped.
To save time, do not solve for $x(t)$.

(e) [10%] Determine the practical resonance frequency ω for the spring-mass equation

$$2x'' + 7x' + 50x = 500\cos(\omega t).$$

Use this page to start your solution.

Chapters 4 and 5

2. (Systems of Differential Equations)

(a) [30%] Display eigenanalysis details for the 3×3 matrix

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix},$$

then display the vector general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(b) [40%] The 3×3 triangular matrix

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix},$$

represents a linear cascade, such as found in brine tank models.

Part 1. Use the linear integrating factor method to find the vector general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Part 2. Explain why the eigenanalysis method fails for this example.

(c) [30%] The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = 5x + 4y, \quad y' = -4x + 5y,$$

which has complex eigenvalues $\lambda = 5 \pm 4i$.

Part 1. Show the details of the method, finally displaying formulas for $x(t), y(t)$.

Part 2. Report a fundamental matrix $\Phi(t)$.

Use this page to start your solution.

Chapter 6

3. **(Linear and Nonlinear Dynamical Systems)**

(a) Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node. Sub-classification into improper or proper node is not required.

$$\vec{u}' = \begin{pmatrix} -3 & 1 \\ -2 & 1 \end{pmatrix} \vec{u}$$

(b) Consider the nonlinear dynamical system

$$\begin{array}{rcl} x' & = & x - 2y^2 + 2y + 32, \\ y' & = & 2x(x + 2y). \end{array}$$

An equilibrium point is $x = -8$, $y = 4$. Compute the Jacobian matrix $A = J(-8, 4)$ of the linearized system at this equilibrium point.

(c) Consider the soft nonlinear spring system $\left\{ \begin{array}{rcl} x' & = & y, \\ y' & = & -5x - 2y + \frac{5}{4}x^3. \end{array} \right.$

At equilibrium point $x = 0$, $y = 0$, the Jacobian matrix is $A = J(0, 0) = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\frac{d}{dt}\vec{u} = A\vec{u}$.

(2) Apply the Pasting Theorem to classify $x = 0$, $y = 0$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

(3) Repeat the classification details of the previous two parts (1), (2) for the other two equilibrium points $(2, 0), (-2, 0)$, for which the Jacobian is $A = J(\pm 2, 0) = \begin{pmatrix} 0 & 1 \\ 10 & -2 \end{pmatrix}$.

Use this page to start your solution.