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# Differential Equations 2280

**Sample Midterm Exam 1**

**Exam Date: Friday, 27 February 2015 at 12:50pm**

**Instructions**: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

   **(a)** [25%] Solve $y' = \dfrac{3+x^2}{1+x^2}$.

   **(b)** [25%] Solve $y' = (2\sin x + \cos x)(\sin x - 2\cos x)$.

   **(c)** [25%] Solve $y' = \dfrac{x\tan\left(\ln(1+x^2)\right)}{1+x^2}$, $y(0) = 2$.

   **(d)** [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(t^2 v(t)) = 0$, $v(2) = 10$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = -20$.

   [Integral tables will be supplied for anything other than basic formulas. This sample problem would require no integral table. The exam problem will be shorter.]

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**2. (Classification of Equations)**
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions $F$ and $G$.

**(a)** [40%] Check ( X ) the problems that can be put into separable form. No details expected.

| | | | |
|---|---|---|---|
| ☐ | $y' + xy = y(2x + e^x) + x^2 y$ | ☐ | $y' = (x - 1)(y + 1) + (1 - x)y$ |
| ☐ | $y' = 2e^{2x-y}e^{3y} + 3e^{3x+2y}$ | ☐ | $y' + x^2 e^y = xy$ |

**(b)** [10%] Is $y' + x(y + 1) = ye^x + x$ separable? No details expected.

**(c)** [10%] Give an example of $y' = f(x, y)$ which is separable and linear but not quadrature. No details expected.

**(d)** [40%] Apply tests to show that $y' = x + e^y$ is not separable and not linear. Supply all details.

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**3. (Solve a Separable Equation)**
Given $(x+3)(y+1)y' = \left((x+3)e^{-x+2} + 3x^2 + 2\right)(y-1)(y+2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for $y$ explicitly and **do not solve** for equilibrium solutions.

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**4. (Linear Equations)**

**(a)** [60%] Solve the linear model $5x'(t) = -160 + \frac{25}{2t+3}x(t)$, $x(0) = 32$. Show all integrating factor steps.

**(b)** [20%] Solve the homogeneous equation $\frac{dy}{dx} - (2x)y = 0$.

**(c)** [20%] Solve $5\frac{dy}{dx} + 10y = 7$ using the superposition principle $y = y_h + y_p$. Expected are answers for $y_h$ and $y_p$.

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**5. (Stability)**

**(a)** [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \left(\ln(1+5x^2)\right)^{1/5} \left(|2x-1|-3\right)^3 (2+x)^2(4-x^2)(1-x^2)^3 e^{\cos x}.$$

Expected in the phase line diagram are equilibrium points and signs of $dx/dt$.

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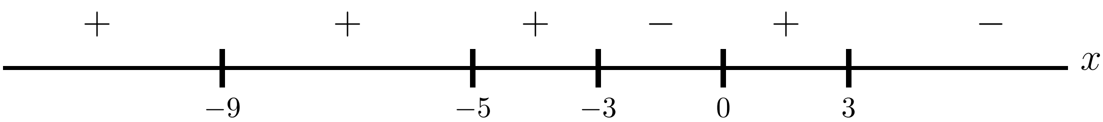
**(b)** [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.

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**6. (ch3)**

Using Euler's theorem on atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c) and (d).

(a) [25%] Find a differential equation $ay'' + by' + cy = 0$ with solutions $2e^{-x}$, $e^{-x} - e^{2x/3}$.

(b) [25%] Solve $y^{(6)} + 4y^{(5)} + 4y^{(4)} = 0$.

(c) [25%] Given characteristic equation $r(r+2)(r^3-4r)^3(r^2+2r+5) = 0$, solve the differential equation.

(d) [25%] Given $4x''(t) + 4x'(t) + 65x(t) = 0$, which represents an unforced damped spring-mass system with $m = 4$, $c = 4$, $k = 65$. Solve the differential equation [15%]. Classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a drawing of the physical model the meaning of constants $m$, $c$, $k$ [5%].

**Notes on Problem 6.**

Part (a)

**1**: $r^2 + r + 1 = 0$, $y = c_1 y_1 + c_2 y_2$, $y_1 = e^{-x/2}\cos(\sqrt{3}x/2)$, $y_2 = e^{-x/2}\sin(\sqrt{3}x/2)$.

**2**: $r^{iv} + 4r^2 = 0$, roots $r = 0, 0, 2i, -2i$. Then $y = (c_1 + c_2 x)e^{0x} + c_3 \cos 2x + c_4 \sin 2x$.

**3**: Write as $(r - a)^2(r + a)^2(r^2 + 16)^3 = 0$ where $a = \sqrt{3}$. Then $y = u_1 e^{ax} + u_2 e^{-ax} + u_3 \cos 4x + u_5 \sin 3x$. The polynomials are $u_1 = c_1 + c_2 x$, $u_2 = c_3 + c_4 x$, $u_3 = c_5 + c_6 x + c_7 x^2$, $u_4 = c_8 + c_9 x + c_{10} x^2$.

Part (b)

Use $4r^2 + 4r + 1 = 0$ and the quadratic formula to obtain roots $r = -1/2, -1/2$. Case 2 of the recipe gives $y = (c_1 + c_2 t)e^{-t/2}$. This is critically damped. The illustration shows a spring, dampener and mass with labels $k$, $c$, $m$, $x$ and the equilibrium position of the mass.

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**7. (ch3)**

(a) [25%] The trial solution $y$ with fewest atoms, according to the method of undetermined coefficients, contains no solution of the homogeneous equation. Explain why, using the example $y'' = 1 + x$.

(b) [75%] Determine for $y^{(4)} + y^{(2)} = x + 2e^x + 3\sin x$ the corrected trial solution for $y_p$ according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients! The trial solution should be the one with fewest atoms.

**Notes on Problem 7.**

The homogeneous solution is $y_h = c_1 + c_2 x + c_3 e^{3x} + c_4 e^{-3x}$, because the characteristic polynomial has roots 0, 0, 3, $-3$.

**1** An initial trial solution $y$ is constructed for atoms 1, $x$, $e^{3x}$, $e^{-3x}$, $\cos x$, $\sin x$ giving

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= (d_1 + d_2 x)e^{3x}, \\ y_2 &= d_3 + d_4 x + d_5 x^2 + d_6 x^3, \\ y_3 &= d_7 e^{-3x}, \\ y_4 &= d_8 \cos x + d_9 \sin x. \end{aligned}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by assignment of constants, all derivatives of the right side of the differential equation.

**2** The fixup rule is applied individually to each of $y_1$, $y_2$, $y_3$, $y_4$ to give the **corrected trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3, \\ y_1 &= x(d_1 + d_2 x)e^{3x}, \\ y_2 &= x^2(d_3 + d_4 x + d_5 x^2 + d_6 x^3), \\ y_3 &= x(d_7 e^{-3x}), \\ y_4 &= d_8 \cos x + d_9 \sin x. \end{aligned}$$

The powers of $x$ multiplied in each case are designed to eliminate terms in the initial trial solution which duplicate atoms appearing in the homogeneous solution $y_h$. The factor is exactly $x^s$ of the Edwards-Penney table, where $s$ is the multiplicity of the characteristic equation root $r$ that produced the related atom in the homogeneous solution $y_h$. By design, unrelated atoms are unaffected by the fixup rule, and that is why $y_4$ was unaltered.

**3** Undetermined coefficient step skipped, according to the problem statement.

**4** Undetermined coefficient step skipped, according to the problem statement.

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