Name ________________________________

# Differential Equations 2280

**Sample Midterm Exam 1**

**Exam Date: Friday, 27 February 2015 at 12:50pm**

**Instructions**: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. The first 5 problems are from a midterm exam in 2009, solutions appended to this PDF. The last two problems have solutions immediately after the problem statement.

1. **(Quadrature Equations)**

   **(a)** [25%] Solve $y' = \dfrac{3+x^2}{1+x^2}$.

   **(b)** [25%] Solve $y' = (2\sin x + \cos x)(\sin x - 2\cos x)$.

   **(c)** [25%] Solve $y' = \dfrac{x\tan\left(\ln(1+x^2)\right)}{1+x^2}$, $y(0) = 2$.

   **(d)** [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(t^2 v(t)) = 0$, $v(2) = 10$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = -20$.

   [Integral tables will be supplied for anything other than basic formulas. This sample problem would require no integral table. The exam problem will be shorter.]

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**2. (Classification of Equations)**
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions $F$ and $G$.

**(a)** [40%] Check ( X ) the problems that can be put into separable form. No details expected.

| | | | |
|---|---|---|---|
| ☐ | $y' + xy = y(2x + e^x) + x^2 y$ | ☐ | $y' = (x-1)(y+1) + (1-x)y$ |
| ☐ | $y' = 2e^{2x-y}e^{3y} + 3e^{3x+2y}$ | ☐ | $y' + x^2 e^y = xy$ |

**(b)** [10%] Is $y' + x(y+1) = ye^x + x$ separable? No details expected.

**(c)** [10%] Give an example of $y' = f(x,y)$ which is separable and linear but not quadrature. No details expected.

**(d)** [40%] Apply tests to show that $y' = x + e^y$ is not separable and not linear. Supply all details.

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**3. (Solve a Separable Equation)**
Given $(x+3)(y+1)y' = \left((x+3)e^{-x+2} + 3x^2 + 2\right)(y-1)(y+2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for $y$ explicitly and **do not solve** for equilibrium solutions.

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**Name.** ________________________________

**4. (Linear Equations)**

**(a)** [60%] Solve the linear model $5x'(t) = -160 + \frac{25}{2t+3}x(t)$, $x(0) = 32$. Show all integrating factor steps.

**(b)** [20%] Solve the homogeneous equation $\frac{dy}{dx} - (2x)y = 0$.

**(c)** [20%] Solve $5\frac{dy}{dx} + 10y = 7$ using the superposition principle $y = y_h + y_p$. Expected are answers for $y_h$ and $y_p$.

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**Name**. ___________________________________

**5. (Stability)**

**(a)** [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \left(\ln(1+5x^2)\right)^{1/5} \left(|2x-1|-3\right)^3 (2+x)^2(4-x^2)(1-x^2)^3 e^{\cos x}.$$

Expected in the phase line diagram are equilibrium points and signs of $dx/dt$.

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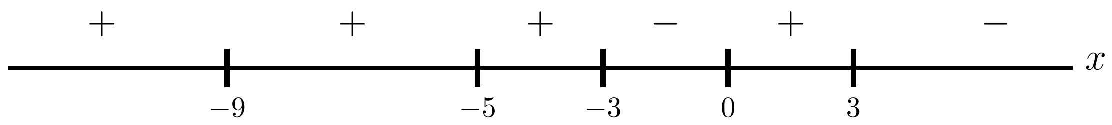
**(b)** [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.

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**Name.** ________________________________

**6. (ch3)**

Using Euler's theorem on atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c) and (d).

(a) [25%] Find a differential equation $ay'' + by' + cy = 0$ with solutions $2e^{-x}$, $e^{-x} - e^{2x/3}$.

(b) [25%] Solve $y^{(6)} + 4y^{(5)} + 4y^{(4)} = 0$.

(c) [25%] Given characteristic equation $r(r+2)(r^3-4r)^3(r^2+2r+5) = 0$, solve the differential equation.

(d) [25%] Given $4x''(t) + 4x'(t) + 65x(t) = 0$, which represents an unforced damped spring-mass system with $m = 4$, $c = 4$, $k = 65$. Solve the differential equation [15%]. Classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a drawing of the physical model the meaning of constants $m$, $c$, $k$ [5%].

**Solution to Problem 6.**

**6(a)**

Divide the first solution by 2. Then Euler atom $e^{-x}$ is a solution, which implies that $r = -1$ is a root of the characteristic equation. Subtract $y_1 = e^{-x}$ and $y_2 = e^{-x} - e^{2x/3}$ to justify that $y = y_1 - y_2 = e^{2x/3}$ is a solution. It is an Euler atom corresponding to root $r = 2/3$. Then the characteristic equation should be $(r-(-1))(r-2/3) = 0$, or $3r^2+r-2 = 0$. The differential equation is $3y'' + y' - 2y = 0$.

**6(b)**

The characteristic equation factors into $r^4(r^2 + 4r + 4) = 0$ with roots $r = 0, 0, 0, 0, -2, -2$. Then $y$ is a linear combination of the Euler atoms $1, x, x^2, x^3, e^{-2x}, xe^{-2x}$.

**6(c)**

The roots of the fully factored equation $r^4(r + 2)^4(r - 2)^3((r + 1)^2 + 4) = 0$ are

$$r = 0, 0, 0, 0, -2, -2, -2, -2, 2, 2, 2, -1 \pm 2i.$$

The solution $y$ is a linear combination of the Euler atoms

$$1, x, x^2, x^3; \quad e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}; \quad e^{2x}, xe^{2x}, x^2e^{2x}; \quad e^{-x}\cos(2x), e^{-x}\sin(2x).$$

**6(d)**

Use $4r^2 + 4r + 65 = 0$ and the quadratic formula to obtain roots $r = -1/2 + 4i, -1/2 - 4i$. Case 2 of the recipe gives $y = (c_1 \cos 4t + c_2 \sin 4t)e^{-t/2}$. This is under-damped (it oscillates). The illustration shows a spring, dashpot and mass with labels $k$, $c$, $m$, $x$ and the equilibrium position of the mass.

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**Name.** ________________________________

**7. (ch3)**

(a) [25%] The trial solution $y$ with fewest Euler solution atoms, according to the method of undetermined coefficients, contains no solution of the homogeneous equation. Explain why, using the example $y'' = 1 + x$.

(b) [75%] Determine for $y^{(4)} + y^{(2)} = x + 2e^x + 3\sin x$ the corrected trial solution for $y_p$ according to the method of undetermined coefficients. **Do not evaluate the undetermined coefficients!** The trial solution should be the one with fewest Euler solution atoms.

**Solution to Problem 7.**

**7(a).** Rule I says the trial solution is $y = d_1 + d_2 x$. Rule II says to multiply by $x$ until no atom is a solution of $y'' = 0$. Then $y = d_1 x^2 + d_2 x^3$ contains no terms of the homogeneous solution $y_h = c_1 + c_2 x$.

**7(b).** The homogeneous solution is $y_h = c_1 + c_2 x + c_3 e^{3x} + c_4 e^{-3x}$, because the characteristic polynomial has roots 0, 0, 3, $-3$.

**1** Rule I constructs an initial trial solution $y$ for Euler solution atoms 1, $x$, $e^{3x}$, $e^{-3x}$, $\cos x$, $\sin x$ giving

$$\begin{array}{rcl} y &=& y_1 + y_2 + y_3 + y_4, \\ y_1 &=& (d_1 + d_2 x)e^{3x}, \\ y_2 &=& d_3 + d_4 x + d_5 x^2 + d_6 x^3, \\ y_3 &=& d_7 e^{-3x}, \\ y_4 &=& d_8 \cos x + d_9 \sin x. \end{array}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by assignment of constants, all derivatives of the right side of the differential equation. Each of $y_1$ to $y_4$ is constructed to have the same **base atom**, which is the Euler atom obtained by stripping the power of $x$. For example, $x^3 = x^3 e^{0x}$ strips to base atom $e^{0x}$ or 1.

**2** Rule II is applied individually to each of $y_1$, $y_2$, $y_3$, $y_4$ to give the **corrected trial solution**

$$\begin{array}{rcl} y &=& y_1 + y_2 + y_3 + y_4, \\ y_1 &=& x(d_1 + d_2 x)e^{3x}, \\ y_2 &=& x^2(d_3 + d_4 x + d_5 x^2 + d_6 x^3), \\ y_3 &=& x(d_7 e^{-3x}), \\ y_4 &=& d_8 \cos x + d_9 \sin x. \end{array}$$

The powers of $x$ multiplied in each case are selected to eliminate terms in the initial trial solution which duplicate homogeneous equation Euler solution atoms. The factor used is exactly $x^s$ of the Edwards-Penney table, where $s$ is the multiplicity of the characteristic equation root $r$ that produced the related atom in the homogeneous solution $y_h$. Terms in $y_4$ are not solutions of the homogeneous equation, therefore $y_4$ is unaltered.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY

# Differential Equations 2280

**Midterm Exam 1 [8:35]**
**Wednesday, 25 February 2009**

**Instructions**: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

**1. (Quadrature Equations)**

**(a)** [25%] Solve $y' = \dfrac{3+x^2}{1+x^2}$.

**(b)** [25%] Solve $y' = (2\sin x + \cos x)(\sin x - 2\cos x)$.

**(c)** [25%] Solve $y' = \dfrac{x\tan(\ln(1+x^2))}{1+x^2}$, $y(0) = 2$.

**(d)** [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(t^2 v(t)) = 0$, $v(2) = 10$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = -20$.

(a) $y = \int \frac{3+x^2}{1+x^2}dx = \int \frac{2\,dx}{1+x^2} + \int 1\,dx = 2\tan^{-1}(x) + x + c$

(b) $y = \int (2\sin x + \cos x)(2\sin x + \cos x)'(-1)\,dx = -\frac{1}{2}(2\sin x + \cos x)^2 + c$

(c) $y = \int \frac{x\tan(\ln(1+x^2))}{1+x^2}dx$ $\qquad u = \ln(1+x^2)$, $du = \frac{2x}{1+x^2}dx$

$= \int \tan(u)\frac{du}{2}$

$= \frac{-1}{2}\ln|\cos(u)| + c$

$= -\frac{1}{2}\ln(\cos(\ln(1+x^2))) + c$

(d) $t^2 v(t) = c \Rightarrow 4v(2) = c \Rightarrow 40 = c$

$\boxed{v(t) = \frac{40}{t^2}}$

$x' = \frac{40}{t^2}$

$x = -40t^{-1} + c \Rightarrow -20 = -40/2 + c \Rightarrow c = 0$

$\boxed{x = -40/t}$

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Name. KEY

**2. (Classification of Equations)**
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions $F$ and $G$.

**(a)** [40%] Check ($\boxed{X}$) the problems that can be put into separable form. No details expected.

| | | | |
|---|---|---|---|
| [X] | $y' + xy = y(2x + e^x) + x^2y$ | [X] | $y' = (x-1)(y+1) + (1-x)y$ |
| [X] | $y' = 2e^{2x-y}e^{3y} + 3e^{3x+2y}$ | [ ] | $y' + x^2e^y = xy$ |

**(b)** [10%] Is $y' + x(y + 1) = ye^x + x$ separable? No details expected.

**(c)** [10%] Give an example of $y' = f(x, y)$ which is separable and linear but not quadrature. No details expected.

**(d)** [40%] Apply tests to show that $y' = x + e^y$ is not separable and not linear. Supply all details.

(a) $y' + xy = 2xy + e^x y + x^2 y$ Linear, Separable
$y' = 2e^{2x}e^{2y} + 3e^{3x}e^{2y}$ Separable
$y' = xy - y + x - 1 + y - xy = x - 1$ SLQ
$y' = -x^2e^y + xy$ Not S, Q or L

(b) $y' = ye^x + x - xy - x = ye^x - xy = y(e^x - x)$
yes, separable.

(c) $y' = xy$

(d) $f(x,y) = x + e^y$
$\frac{f_x}{f} = \frac{1}{x + e^y}$ not indep of $y$ $\Rightarrow$ not separable
$f_y = e^y$ not indep of $y$ $\Rightarrow$ Not linear

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Name. KEY

**3. (Solve a Separable Equation)**
Given $(x+3)(y+1)y' = \left((x+3)e^{-x+2} + 3x^2 + 2\right)(y-1)(y+2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for $y$ explicitly and **do not solve** for equilibrium solutions.

$$\frac{y+1}{(y-1)(y+2)}y' = e^{2-x} + \frac{3x^2+2}{x+3}$$

$$\left(\frac{A}{y-1} + \frac{B}{y+2}\right)y' = e^{2-x} + 3x - 9 + \frac{29}{x+3}$$

integrate

$$\frac{2}{3}\ln|y-1| + \frac{1}{3}\ln|y+2| = -e^{-x+2} + \frac{3}{2}x^2 - 9x + 29\ln|x+3| + C$$

Long Division

$$\begin{array}{r} 3x - 9 \\ x+3 \overline{)\, 3x^2 + 2} \\ 3x^2 + 9x \\ \hline -9x + 2 \\ -9x - 27 \\ \hline 29 \end{array}$$

partial fractions

$y+1 = A(y+2) + B(y-1)$

$-1 = -3B$

$2 = 3A$

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Name. KEY

**4. (Linear Equations)**

**(a)** [60%] Solve the linear model $5x'(t) = -160 + \frac{25}{2t+3}x(t)$, $x(0) = 32$. Show all integrating factor steps.

**(b)** [20%] Solve the homogeneous equation $\frac{dy}{dx} - (2x)y = 0$.

**(c)** [20%] Solve $5\frac{dy}{dx} + 10y = 7$ using the superposition principle $y = y_h + y_p$. Expected are answers for $y_h$ and $y_p$.

(a) $x' + \frac{-5}{2t+3} = \frac{-160}{5}$, $x(0) = 32$

$u = \int \frac{-5}{2t+3}\,dt$

$u = -\frac{5}{2}\ln|2t+3|$

$e^u = (2t+3)^{-5/2}$

$(e^u x)' = -32e^u$

$e^u x = -32\int (2t+3)^{-5/2}\,dt$

$= -32\,\frac{(2t+3)^{-3/2}}{(-3/2)(2)} + c$

$x = \frac{32}{3}(2t+3) + c\,(2t+3)^{5/2} \;\rightarrow\; 32 = \frac{32}{3}(0+3) + c\,3^{5/2}$

$\rightarrow\; c = 0$

$$\boxed{x = \frac{64}{3}t + 32}$$

(b) $y = \frac{c}{e^{-x^2}}$

(c) $y = \frac{7}{10} + \frac{c}{e^{2x}}$

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Name. KEY

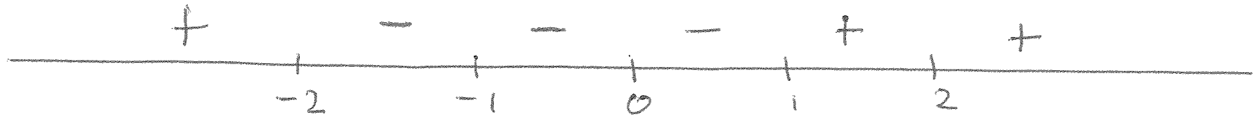
**5. (Stability)**

**(a)** [50%] Draw a phase line diagram for the differential equation

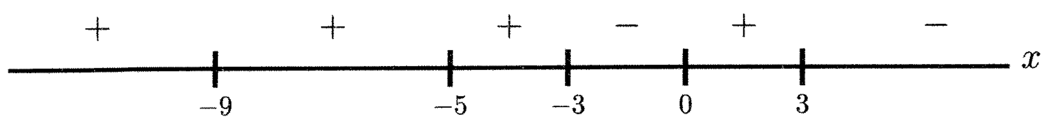
$$\frac{dx}{dt} = \left(\ln(1+5x^2)\right)^{1/5}\left(|2x-1|-3\right)^3(2+x)^2(4-x^2)(1-x^2)^3 e^{\cos x}.$$

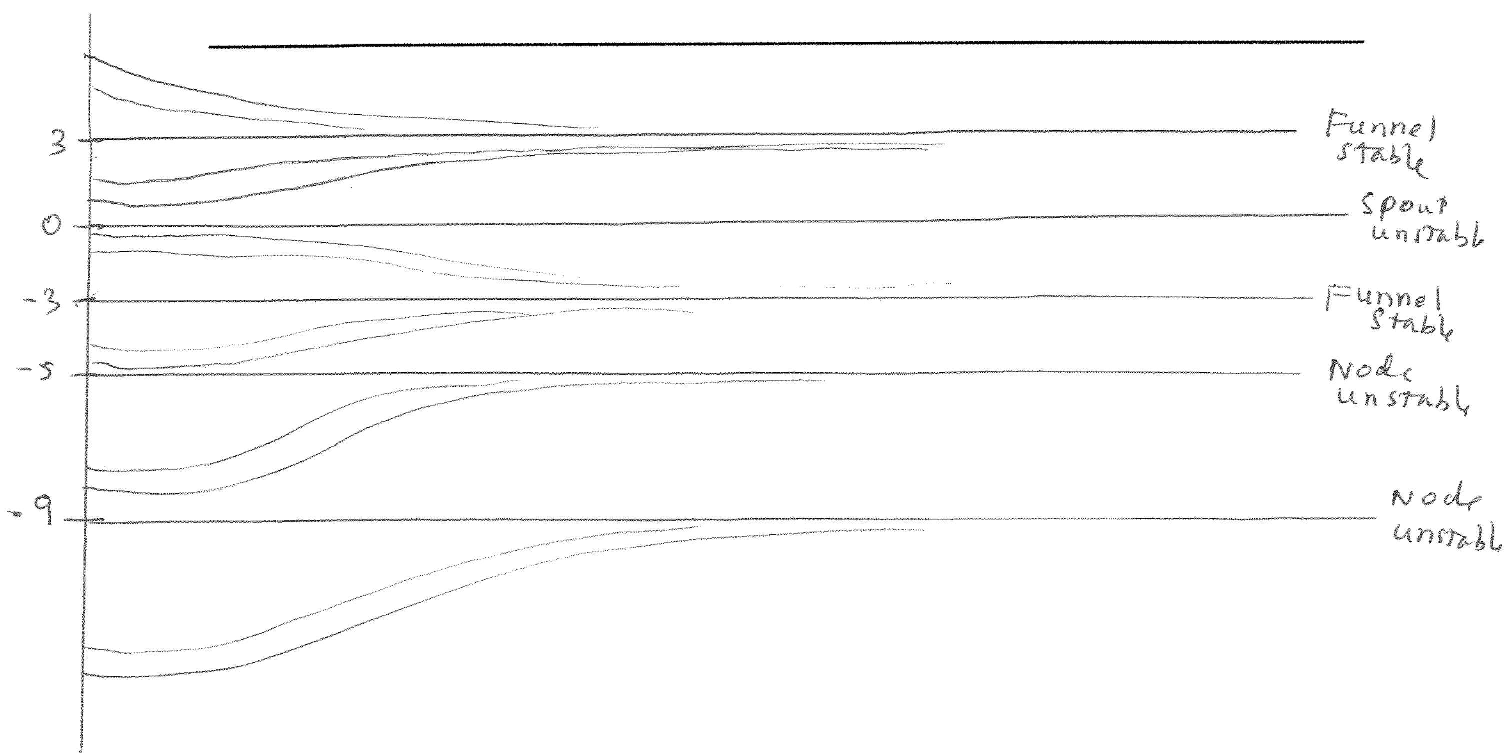
$x = 0$
$2x-1-3=0$
$2x-1+3=0$
$x+2=0$
$x-2=0$
$x=1$
$x=-1$

Expected in the phase line diagram are equilibrium points and signs of $dx/dt$.




**(b)** [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.

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