Name ________________________________

Differential Equations 2280

Sample Midterm Exam 1

Exam Date: Friday, 27 February 2015 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. The first 5 problems are from a midterm exam in 2009, solutions appended to this PDF. The last two problems have solutions immediately after the problem statement. Last edit 23 Feb.

1. **(Quadrature Equations)**

   (a) [25%] Solve $y' = \dfrac{3+x^2}{1+x^2}$.

   (b) [25%] Solve $y' = (2\sin x + \cos x)(\sin x - 2\cos x)$.

   (c) [25%] Solve $y' = \dfrac{x\tan\left(\ln(1+x^2)\right)}{1+x^2}$, $y(0) = 2$.

   (d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(t^2 v(t)) = 0$, $v(2) = 10$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = -20$.

   [Integral tables will be supplied for anything other than basic formulas. This sample problem would require no integral table. The exam problem will be shorter.]

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2. (Classification of Equations)
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check (X) the problems that can be put into separable form. No details expected.

☐	$y' + xy = y(2x + e^x) + x^2 y$	☐	$y' = (x-1)(y+1) + (1-x)y$
☐	$y' = 2e^{2x-y}e^{3y} + 3e^{3x+2y}$	☐	$y' + x^2 e^y = xy$

(b) [10%] Is $y' + x(y+1) = ye^x + x$ separable? No details expected.

(c) [10%] Give an example of $y' = f(x,y)$ which is separable and linear but not quadrature. No details expected.

(d) [40%] Apply tests to show that $y' = x + e^y$ is not separable and not linear. Supply all details.

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3. (Solve a Separable Equation)
Given $(x+3)(y+1)y' = \left((x+3)e^{-x+2} + 3x^2 + 2\right)(y-1)(y+2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

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4. (Linear Equations)

(a) [60%] Solve the linear model $5x'(t) = -160 + \frac{25}{2t+3}x(t)$, $x(0) = 32$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\frac{dy}{dx} - (2x)y = 0$.

(c) [20%] Solve $5\frac{dy}{dx} + 10y = 7$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

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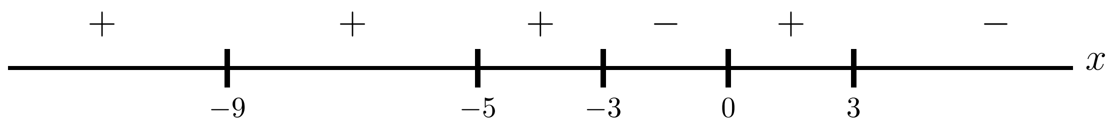
5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \left(\ln(1+5x^2)\right)^{1/5}\left(|2x-1|-3\right)^3(2+x)^2(4-x^2)(1-x^2)^3 e^{\cos x}.$$

Expected in the phase line diagram are equilibrium points and signs of dx/dt.

(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.



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6. (ch3)

Using Euler's theorem on atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c) and (d).

(a) [25%] Find a differential equation $ay'' + by' + cy = 0$ with solutions $2e^{-x}$, $e^{-x} - e^{2x/3}$.

(b) [25%] Solve $y^{(6)} + 4y^{(5)} + 4y^{(4)} = 0$.

(c) [25%] Given characteristic equation $r(r+2)(r^3-4r)^3(r^2+2r+5) = 0$, solve the differential equation.

(d) [25%] Given $4x''(t) + 4x'(t) + 65x(t) = 0$, which represents an unforced damped spring-mass system with $m = 4$, $c = 4$, $k = 65$. Solve the differential equation [15%]. Classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a drawing of the physical model the meaning of constants m, c, k [5%].

Solution to Problem 6.

6(a)

Divide the first solution by 2. Then Euler atom e^{-x} is a solution, which implies that $r = -1$ is a root of the characteristic equation. Subtract $y_1 = e^{-x}$ and $y_2 = e^{-x} - e^{2x/3}$ to justify that $y = y_1 - y_2 = e^{2x/3}$ is a solution. It is an Euler atom corresponding to root $r = 2/3$. Then the characteristic equation should be $(r-(-1))(r-2/3) = 0$, or $3r^2+r-2 = 0$. The differential equation is $3y'' + y' - 2y = 0$.

6(b)

The characteristic equation factors into $r^4(r^2 + 4r + 4) = 0$ with roots $r = 0, 0, 0, 0, -2, -2$. Then y is a linear combination of the Euler atoms $1, x, x^2, x^3, e^{-2x}, xe^{-2x}$.

6(c)

The roots of the fully factored equation $r^4(r + 2)^4(r - 2)^3((r + 1)^2 + 4) = 0$ are

$$r = 0, 0, 0, 0, -2, -2, -2, -2, 2, 2, 2, -1 \pm 2i.$$

The solution y is a linear combination of the Euler atoms

$$1, x, x^2, x^3; \quad e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}; \quad e^{2x}, xe^{2x}, x^2e^{2x}; \quad e^{-x}\cos(2x), e^{-x}\sin(2x).$$

6(d)

Use $4r^2 + 4r + 65 = 0$ and the quadratic formula to obtain roots $r = -1/2 + 4i, -1/2 - 4i$. Case 2 of the recipe gives $y = (c_1 \cos 4t + c_2 \sin 4t)e^{-t/2}$. This is under-damped (it oscillates). The illustration shows a spring, dashpot and mass with labels k, c, m, x and the equilibrium position of the mass.

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7. (ch3)
(a) [25%] The trial solution y with fewest Euler solution atoms, according to the method of undetermined coefficients, contains no solution of the homogeneous equation. Explain why, using the example $y'' = 1 + x$.

(b) [75%] Determine for $y^{(4)} + y^{(2)} = x + 2e^x + 3\sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate the undetermined coefficients!** The trial solution should be the one with fewest Euler solution atoms.

Solution to Problem 7.
7(a). Rule I says the trial solution is $y = d_1 + d_2 x$. Rule II says to multiply by x until no atom is a solution of $y'' = 0$. Then $y = d_1 x^2 + d_2 x^3$ contains no terms of the homogeneous solution $y_h = c_1 + c_2 x$.

7(b). The homogeneous equation $y^{(4)} + y^{(2)} = 0$ has solution $y_h = c_1 + c_2 x + c_3 \cos x + c_4 \sin x$, because the characteristic polynomial has roots 0, 0, i, $-i$.

1 Rule I constructs an initial trial solution y from the list of independent Euler atoms

$$e^x, \quad 1, \quad x, \quad \cos x, \quad \sin x.$$

Linear combinations of these atoms are supposed to reproduce, by assignment of constants, all derivatives of $F(x) = x + 2e^x + 3\sin x$, which is the right side of the differential equation. Each of y_1 to y_4 in the display below is constructed to have the same **base atom**, which is the Euler atom obtained by stripping the power of x. For example, $x = xe^{0x}$ strips to base atom e^{0x} or 1.

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 e^x, \\ y_2 &= d_2 + d_3 x, \\ y_3 &= d_4 \cos x, \\ y_4 &= d_5 \sin x. \end{aligned}$$

2 Rule II is applied individually to each of y_1, y_2, y_3, y_4 to give the **corrected trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 e^x, \\ y_2 &= x^2(d_2 + d_3 x), \\ y_3 &= x(d_4 \cos x), \\ y_4 &= x(d_5 \sin x). \end{aligned}$$

The powers of x multiplied in each case are selected to eliminate terms in the initial trial solution which duplicate homogeneous equation Euler solution atoms. The factor used is exactly x^s of the Edwards-Penney table, where s is the multiplicity of the characteristic equation root r that produced the related atom in the homogeneous solution y_h. The atom in y_1 is not a solution of the homogeneous equation, therefore y_1 is unaltered.

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