Name ________________________________

Differential Equations 2280

Midterm Exam 1

Exam Date: Friday, 27 February 2015 at 12:50pm

Edited Exam 1 with Solutions

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

   (a) [40%] Solve $y' = \dfrac{3+x^2}{2+x}$.

   (b) [60%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(e^t v(t)) = 2\,e^{2t}$, $v(0) = 5$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = 2$.

   Solution to Problem 1.

   (a) Answer $y\,(x) = \frac{1}{2}\,x^2 - 2\,x + 7\,\ln(2+x) + c$. The integral of $F(x) = \frac{3+x^2}{2+x}$ is found by substitution $u = 2 + x$, resulting in the new integration problem $\int F\,dx = \int (u - 4 + 7/u)du$.

   (b) Velocity $v(t) = e^t + 4e^{-t}$ by quadrature. Integrate $x'(t) = e^t + 4e^{-t}$ with $x(0) = 2$ to obtain position $x(t) = e^t - 4e^{-t} + 5$.

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2. (Classification of Equations)
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [40%] The equation $y' + x(y+3) = ye^x + 3x$ is separable. Provide formulas for $F(x)$ and $G(y)$.

(b) [60%] Apply partial derivative tests to show that $y' = x + y$ is linear but not separable. Supply all details.

Solution to Problem 2.
(a) The equation is $y' = ye^x - xy = (e^x - x)y$. Then $F(x) = e^x - x$, $G(y) = y$.

(b) Let $f(x,y) = x + y$. Then $\partial f/\partial y = 1$, which is independent of y, hence the equation $y' = f(x,y)$ is linear. The negative test is $\frac{\partial f/\partial y}{f}$ depends on x. In this case, the fraction is

$$\frac{\partial f/\partial y}{f} = \frac{1}{f} = \frac{1}{x+y}.$$

At $y = 0$, this reduces to $1/x$, which depends on x, therefore the equation $y' = f(x,y)$ is not separable.

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3. (Solve a Separable Equation)
Given $(5y+10)y' = \left(xe^{-x} + \sin(x)\cos(x)\right)\left(y^2+3y-4\right)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

Solution to Problem 3.
The solution by separation of variables identifies the separated equation $y' = F(x)G(y)$ using

$$F(x) = xe^{-x} + \sin(x)\cos(x), \quad G(y) = \frac{y^2+3y-4}{5y+10}.$$

The integral of F is done by parts and also by u-substitution.

$$\begin{array}{lll} \int F dx & = & \int xe^{-x}\,dx + \int \sin(x)\cos(x)\,dx \\ & = & I_1 + I_2. \\ I_1 & = & \int xe^{-x}\,dx \\ & = & -xe^{-x} - \int e^{-x}\,dx, \quad \text{parts} \quad u = x, dv = e^{-x}dx, \\ & = & xe{-}x - e^{-x} + c_1 \\ I_2 & = & \int \sin(x)\cos(x)\,dx \\ & = & \int u\,du, \quad u = \sin(x), du = \cos(x)dx, \\ & = & u^2/2 + c_2 \\ & = & \frac{1}{2}\sin^2(x) + c_2 \end{array}$$

Then $\displaystyle\int F dx = xe{-}x - e^{-x} + \frac{1}{2}\sin^2(x) + c_3$.

The integral of $1/G(y)$ requires partial fractions. The details:

$$\begin{array}{lll} \displaystyle\int \frac{dx}{G(y(x))} & = & \displaystyle\int \frac{5u+10}{u^2+3u-4}\,du, \quad u = y(x), du = y'(x)dx, \\ & = & \displaystyle\int \frac{5u+10}{(u+4)(u-1)}\,du \\ & = & \displaystyle\int \frac{A}{u+4} + \frac{B}{u-1}\,du, \quad A, B \quad \text{determined later}, \\ & = & A\ln|u+4| + B\ln|u-1| + c_4 \end{array}$$

The partial fraction problem

$$\frac{5u+10}{(u+4)(u-1)} = \frac{A}{u+4} + \frac{B}{u-1}$$

can be solved in a variety of ways, with answer $A = \frac{-20+10}{-5} = 2$ and $B = \frac{15}{5} = 3$. The final implicit solution is obtained from $\int \frac{dx}{G(y(x))} = \int F(x)dx$, which gives the equation

$$2\ln|y+4| + 3\ln|y-1| = xe{-}x - e^{-x} + \frac{1}{2}\sin^2(x) + c.$$

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4. (Linear Equations)

(a) [60%] Solve the linear model $2x'(t) = -64 + \frac{10}{3t+2}x(t)$, $x(0) = 32$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\frac{dy}{dx} - (\cos(x))y = 0$.

(c) [20%] Solve $5\frac{dy}{dx} - 7y = 10$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

Solution to Problem 4.

(a) The answer is $v(t) = 32 + 48t$. The details:

$$\begin{aligned}
&v'(t) = -32 + \frac{5}{3t+2}v(t), \\
&v'(t) + \frac{-5}{3t+2}v(t) = -32, \quad \text{standard form } v' + p(t)v = q(t) \\
&p(t) = \frac{-5}{3t+2}, \\
&W = e^{\int p\,dt}, \quad \text{integrating factor} \\
&W = e^u, \quad u = \textstyle\int p\,dt = -\frac{5}{3}\ln|3t+2| = \ln\left(|3t+2|^{-5/3}\right) \\
&W = (3t+2)^{-5/3}, \quad \text{Final choice for } W.
\end{aligned}$$

Then replace the left side of $v' + pv = q$ by $(vW)'/W$ to obtain

$$\begin{aligned}
&v'(t) + \frac{-5}{3t+2}v(t) = -32, \quad \text{standard form } v' + p(t)v = q(t) \\
&\frac{(vW)'}{W} = -32, \quad \text{Replace left side by quotient } (vW)'/W \\
&(vW)' = -32W, \quad \text{cross-multiply} \\
&vW = -32\textstyle\int W\,dt, \quad \text{quadrature step.}
\end{aligned}$$

The evaluation of the integral is from the power rule:

$$\int -32W\,dt = -32\int (3t+2)^{-5/3}dt = -32\frac{(3t+2)^{-2/3}}{(-2/3)(3)} + c.$$

Division by $W = (3t+2)^{-5/3}$ then gives the general solution

$$v(t) = \frac{c}{W} - \frac{32}{-2}(3t+2)^{-2/3}(3t+2)^{5/3}.$$

Constant c evaluates to $c = 0$ because of initial condition $v(0) = 32$. Then

$$v(t) = \frac{32}{-2}(3t+2)^{-2/3}(3t+2)^{5/3} = 16(3t+2)^{-\frac{2}{3}+\frac{5}{3}} = 16(3t+2).$$

(b) The answer is y = constant divided by the integrating factor: $y = \frac{c}{W}$. Because $W = e^u$ where $u = \int -\cos(x)dx = -\sin x$, then $y = ce^{\sin x}$.

(c) The equilibrium solution (a constant solution) is $y_p = -\frac{10}{7}$. The homogeneous solution is $y_h = ce^{7x/5}$ = constant divided by the integrating factor. Then $y = y_p + y_h = -\frac{10}{7} + ce^{7x/5}$.

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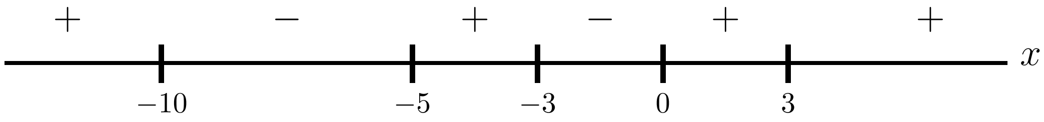
5. (Stability)

Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.



Solution to Problem 5.
The graphic is drawn using increasing and decreasing curves, which may or may not be depicted with turning points. The rules:

1. A curve drawn between equilibria is increasing if the sign is PLUS.
2. A curve drawn between equilibria is decreasing if the sign is MINUS.
3. Label: FUNNEL, STABLE
   The signs left to right are PLUS MINUS crossing the equilibrium point.
4. Label: SPOUT, UNSTABLE
   The signs left to right are MINUS PLUS crossing the equilibrium point.
5. Label: NODE, UNSTABLE
   The signs left to right are PLUS PLUS crossing the equilibrium point, or
   The signs left to right are MINUS MINUS crossing the equilibrium point.

The answer:

$x = -10$: FUNNEL, STABLE
$x = -5$: SPOUT, UNSTABLE
$x = -3$: FUNNEL, STABLE
$x = 0$: SPOUT, UNSTABLE
$x = 3$: NODE, STABLE

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6. (ch3)
Using Euler's theorem on atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c).

(a) [40%] Find a differential equation $ay'' + by' + cy = 0$ which has particular solutions $-5e^{-x} + xe^{-x}$, $10e^{-x} + xe^{-x}$.

(b) [30%] Given characteristic equation $r(r-2)(r^3+4r)^4(r^2+2r+17) = 0$, solve the differential equation.

(c) [30%] Given $mx''(t) + cx'(t) + kx(t) = 0$, which represents an unforced damped spring-mass system. Assume $m = 4$, $c = 4$, $k = 129$. Classify the answer as over-damped, critically damped or under-damped. Illustrate in a drawing the assignment of physical constants m, c, k and the initial conditions $x(0) = 0$, $x'(0) = 1$.

Solution to Problem 6.

6(a)
Divide the first solution by 2. Then Euler atom e^{-x} is a solution, which implies that $r = -1$ is a root of the characteristic equation. Subtract $y_1 = e^{-x}$ and $y_2 = e^{-x} - e^{2x/3}$ to justify that $y = y_1 - y_2 = e^{2x/3}$ is a solution. It is an Euler atom corresponding to root $r = 2/3$. Then the characteristic equation should be $(r-(-1))(r-2/3) = 0$, or $3r^2+r-2 = 0$. The differential equation is $3y'' + y' - 2y = 0$.

6(b)
The characteristic equation factors into $r^4(r^2 + 4r + 4) = 0$ with roots $r = 0, 0, 0, 0, -2, -2$. Then y is a linear combination of the Euler atoms $1, x, x^2, x^3, e^{-2x}, xe^{-2x}$.

6(c)
The roots of the fully factored equation $r^4(r+2)^4(r-2)^3((r+1)^2+4) = 0$ are

$$r = 0, 0, 0, 0, -2, -2, -2, -2, 2, 2, 2, -1 \pm 2i.$$

The solution y is a linear combination of the Euler atoms

$$1, x, x^2, x^3; \quad e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}; \quad e^{2x}, xe^{2x}, x^2e^{2x}; \quad e^{-x}\cos(2x), e^{-x}\sin(2x).$$

6(d)
Use $4r^2 + 4r + 65 = 0$ and the quadratic formula to obtain roots $r = -1/2 + 4i, -1/2 - 4i$ and Euler atoms $\cos 4t, \sin 4t$. Then $y = (c_1 \cos 4t + c_2 \sin 4t)e^{-t/2}$. This is under-damped (it oscillates). The illustration shows a spring, a dashpot and a mass with labels k, c, m. Also shown is mass elongation x and the equilibrium position $x = 0$.

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7. (ch3)

Determine for $y^{(4)}+y^{(2)} = x+2e^x+3\sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate the undetermined coefficients!** The trial solution should be the one with fewest Euler solution atoms.

Solution to Problem 7.
The homogeneous equation $y^{(4)} + y^{(2)} = 0$ has solution $y_h = c_1 + c_2 x + c_3 \cos x + c_4 \sin x$, because the characteristic polynomial has roots 0, 0, i, $-i$.

1 Rule I constructs an initial trial solution y from the list of independent Euler atoms

$$e^x, \quad 1, \quad x, \quad \cos x, \quad \sin x.$$

Linear combinations of these atoms are supposed to reproduce, by assignment of constants, all derivatives of $F(x) = x + 2e^x + 3\sin x$, which is the right side of the differential equation. Each of y_1 to y_4 in the display below is constructed to have the same **base atom**, which is the Euler atom obtained by stripping the power of x. For example, $x = xe^{0x}$ strips to base atom e^{0x} or 1.

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 e^x, \\ y_2 &= d_2 + d_3 x, \\ y_3 &= d_4 \cos x, \\ y_4 &= d_5 \sin x. \end{aligned}$$

2 Rule II is applied individually to each of y_1, y_2, y_3, y_4 to give the **corrected trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 e^x, \\ y_2 &= x^2(d_2 + d_3 x), \\ y_3 &= x(d_4 \cos x), \\ y_4 &= x(d_5 \sin x). \end{aligned}$$

The powers of x multiplied in each case are selected to eliminate terms in the initial trial solution which duplicate homogeneous equation Euler solution atoms. The factor used is exactly x^s of the Edwards-Penney table, where s is the multiplicity of the characteristic equation root r that produced the related atom in the homogeneous solution y_h. The atom in y_1 is not a solution of the homogeneous equation, therefore y_1 is unaltered.

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