Differential Equations 2280
Midterm Exam 3
Wednesday, 29 April 2009

Instructions: This in-class exam is 50 minutes. Do four problems only. The first problem was done on 22 April. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. Each problem is scored 100.

Please discard this sheet after reading it.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. ________________________ Midterm 3 Ver 2, 2280 8:35

1. (ch7) Do enough to make 100%
(1a) [50%] Derive the formula $\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))|_{s\to(s-a)}$.
(1b) [50%] Solve $x''' + x'' = 0$, $x(0) = -1$, $x'(0) = 1$, $x''(0) = 0$ by Laplace's Method.
(1c) [50%] The system $x' = x + y$, $y' = -y + 2$, $x(0) = 0$, $y(0) = 0$ has unique solution $x(t) = -2 + e^t + e^{-t}$, $y(t) = 2 - 2e^{-t}$. Show the Laplace Method details.

Answer: (1a) $\mathcal{L}(e^{at}f(t)) = \int_0^\infty f(t)e^{at-st}dt = \int_0^\infty f(t)e^{-ut}dt$ where $u = s - a$. The last integral is $\mathcal{L}(f(t))$ with s replaced by $s - a$.
(1b) $L(x) = \frac{-s^2+1}{s^3+s^2} = \frac{1-s}{s^2} = \frac{1}{s^2} - \frac{1}{s} = \mathcal{L}(t-1)$ implies $x(t) = t - 1$.
(1c) Transform the equations with $\mathcal{L}$ and collect into a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$. A shortcut is Laplace's resolvent method. Then

$$\begin{pmatrix} s-1 & -1 \\ 0 & s+1 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 2/s \end{pmatrix}.$$

Solve by Cramer's rule to obtain $\mathcal{L}(x) = \frac{2}{s(s-1)(s+1)}$, $\mathcal{L}(y) = \frac{2}{s(s+1)}$. Then partial fractions and the backward Laplace table imply $x(t) = -2 + e^t + e^{-t}$, $y(t) = 2 - 2e^{-t}$.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. __________________________ Midterm 3 Ver 2, 2280 8:35

2. (ch5) Do both
The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t} + c_4\mathbf{v}_4 e^{\lambda_4 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3, 4$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 5 & 1 & 1 & 0 \\ 1 & 5 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

then
(2a) [75%] Display eigenanalysis details for A.
(2b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Answer: (2a) Use cofactor expansion on the last row of $\det(A - \lambda I)$ to obtain the expansion $(4-\lambda)^2(4-\lambda)(6-\lambda)$. Then $\lambda = 4, 4, 4, 6$. Two sequences of **rref** computations are required on augmented matrices constructed from $A-4I$, $A-6I$ to find the eigenpairs

$$\left(4, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right), \quad \left(6, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right).$$

(2b) $\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{4t}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_4 e^{6t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. ________________________ Midterm 3 Ver 2, 2280 8:35

3. (ch5) Do enough to make 100%

(3a) [50%] The eigenvalues are 4, 6 for the matrix $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$. Show details from either the eigenanalysis method or the Laplace method.

(3b) [50%] Using the same matrix A from part (3a), display the solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton Method. To save time, write out the system to be solved for the two vectors, and then stop, without solving for the vectors.

(3c) [50%] Using the same matrix A from part (3a), compute the exponential matrix e^{At} by any known method, for example, the formula $e^{At} = \Phi(t)\Phi^{-1}(0)$, or Putzer's formula.

Answer: (3a) The eigenpairs of A are

$$\left(4, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \quad \left(6, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

which implies the eigenanalysis general solution

$$\mathbf{u}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(3b) $\mathbf{u}(t) = e^{4t}\vec{\mathbf{c}}_1 + e^{6t}\vec{\mathbf{c}}_2$. Differentiate once and use $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, then set $t = 0$. The resulting system is

$$\begin{aligned} \vec{\mathbf{u}}_0 &= e^0\vec{\mathbf{c}}_1 + e^0\vec{\mathbf{c}}_2 \\ A\vec{\mathbf{u}}_0 &= 4e^0\vec{\mathbf{c}}_1 + 6e^0\vec{\mathbf{c}}_2 \end{aligned}$$

(3c) Putzer's result is $e^{At} = e^{4t}I + \frac{e^{4t}-e^{6t}}{4-6}(A - 4I)$. Functions r_1, r_2 are computed from $r_1' = 4r_1$, $r_1(0) = 1$, $r_2' = 6r_2 + r_1$, $r_2(0) = 0$.

$$e^{At} = \frac{1}{2} \begin{pmatrix} e^{4t} + e^{6t} & e^{6t} - e^{4t} \\ e^{6t} - e^{4t} & e^{4t} + e^{6t} \end{pmatrix}.$$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. ________________________ Midterm 3 Ver 2, 2280 8:35

4. (ch5) Do both

(4a) [50%] Display the solution of $\mathbf{u}' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, using any method that applies.

(4b) [50%] Display the variation of parameters formula for the system below. Then integrate to find $\mathbf{u}_p(t)$ for $\mathbf{u}' = A\mathbf{u}$.

$$\mathbf{u}' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}.$$

Answer: (4a) **Resolvent method**. The resolvent equation $(sI - A)\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}(0)$ is the system

$$\begin{pmatrix} s-2 & 0 \\ -1 & s-2 \end{pmatrix}\begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The system is solved by Cramer's rule for unknowns $\mathcal{L}(x)$, $\mathcal{L}(y)$ to obtain

$$\mathcal{L}(x) = \frac{0}{(s-2)^2}, \quad \mathcal{L}(y) = \frac{s-2}{(s-2)^2}.$$

The backward Laplace table implies

$$x(t) = 0, \quad y(t) = e^{2t}.$$

Best method. Look at the equations as scalar equations $x' = 2x$, $x(0) = 0$ and $y' = x + 2y$, $y(0) = 1$. Clearly $x(t) = 0$ and then $y' = 0 + 2y$, $y(0) = 1$ implies $y(t) = e^{2t}$.

(4b) Putzer's formula gives

$$e^{At} = e^{2t}I + te^{2t}(A - 2I) = \begin{pmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{pmatrix}.$$

Then $\vec{\mathbf{u}}_p(t) = e^{At}\int_0^t e^{-Au}\begin{pmatrix} e^{2u} \\ 0 \end{pmatrix}du = e^{At}\int_0^t \begin{pmatrix} 1 \\ -u \end{pmatrix}du = \begin{pmatrix} te^{2t} \\ t^2e^{2t}/2 \end{pmatrix}$.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. ________________________ Midterm 3 Ver 2, 2280 8:35

5. (ch6) Do enough to make 100%

(5a) [25%] Which of the four types *center, spiral, node, saddle* can be unstable at $t = \infty$? Explain your answer.

(5b) [25%] Give an example of a linear 2-dimensional system $\mathbf{u}' = A\mathbf{u}$ with a saddle at equilibrium point $x = y = 0$, and A is not triangular.

(5c) [25%] Give an example of a nonlinear 2-dimensional predator-prey system with exactly four equilibria.

(5d) [25%] Display a formula for the general solution of the equation $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{u}$. Then explain why the system has a spiral at $(0,0)$.

(5e) [25%] Is the origin an isolated equilibrium point of the $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{u}$? Explain your answer.

Answer: (5a) All except the center, which is stable but not asymptotically stable. All the others correspond to a general solution which can have an exponential factor e^{kt} in each term. If $k > 0$, then the solution cannot approach the origin at $t = \infty$.

(5b) Required are characteristic roots like 1, -1. Let $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Define $A = PBP^{-1}$ where $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\mathbf{u}' = A\mathbf{u}$ has a saddle at the origin, because the characteristic roots of A are still $1, -1$. And $A = \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$ is not triangular.

(5c) **Example**: The nonlinear predator-prey system $x' = (x+y-4)x$, $y' = (-x+2y-2)y$ has exactly four equilibrium points $(0,0)$, $(4,0)$, $(0,1)$, $(2,2)$.

(5d) The characteristic equation $\det(A - \lambda I) = 0$ is $(1-\lambda)^2 + 1 = 0$ with complex roots $1 \pm i$ and corresponding atoms $e^t \cos t$, $e^t \sin t$. Then the Cayley-Hamilton Method implies

$$\vec{\mathbf{u}}(t) = e^t \cos t \vec{\mathbf{c}}_1 + e^t \sin t \vec{\mathbf{c}}_2.$$

Explanation, why the classification is a spiral. Such solutions containing sine and cosine factors wrap around the origin. This makes it a spiral or a center. Because of the exponential factor e^t, it is asymptotically stable at $t = -\infty$, which disallows a center, so it is a spiral.

(5e) No, because $\det(A) = 0$. In this case, $A\mathbf{u} = \mathbf{0}$ has infinitely many solutions, describing a line of equilibria through the origin. This implies the equilibrium point $(0,0)$ is not isolated [you cannot draw a circle about $(0,0)$ which contains no other equilibrium point].

Use this page to start your solution. Attach extra pages as needed, then staple.