Differential Equations 2280
Midterm Exam 2
Wednesday, 25 March 2008

Instructions: This in-class exam is 15 minutes. No calculators, notes, tables or books.

Do one problem only.

No answer check is expected. Details count 75%. The answer counts 25%.

Errata: 1(b) $y^{(3)}$ should be $y^{(4)}$
4(c) Label is 4(e), should be 4(c)

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Name. KEY

S2009 Midterm 2, 2280 8:35

1. (ch3)
Using Euler's theorem on atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c) and (d).

(a) [25%] Find a differential equation $ay''+by'+cy=0$ with solutions $2e^{-x}$, $e^{-x}-e^{2x/3}$.

(b) [25%] Solve $y^{(6)}+4y^{(5)}+4y^{(4)}=0$.

(c) [25%] Given characteristic equation $r(r+2)(r^3-4r)^3(r^2+2r+5)=0$, solve the differential equation.

(d) [25%] Given $4x''(t)+4x'(t)+65x(t)=0$, which represents an unforced damped spring-mass system with $m=4$, $c=4$, $k=65$. Solve the differential equation [15%]. Classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a drawing of the physical model the meaning of constants m, c, k [5%].

(a) roots $= -1,\ 2/3 \Rightarrow (r+1)(r-2/3)=0 \Rightarrow (r+1)(3r-2)=0$
$\Rightarrow 3r^2+r-2=0 \Rightarrow 3y''+y'-2y=0$

(b) $r^4(r^2+4r+4)=0$, $r=0,0,0,-2,-2$
$y = c_1+c_2x+c_3x^2+c_4e^{-2x}+c_5xe^{-2x}$

(c) $r(r+2)(r-2)^3(r+2)^3r^3((r+1)^2+4)=0$
$r^4(r+2)^4(r-2)^3((r+1)^2+4)=0$
atoms $= 1, x, x^2, x^3, e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}, e^{2x}, xe^{2x}, e^{-x}\cos 2x,$
$e^{-x}\sin 2x$
y = linear combination of the atoms above (12 atoms)

(d) $4r^2+4r+65=0 \Rightarrow 4(r+\frac{1}{2})^2+64=0 \Rightarrow (r+\frac{1}{2})^2+16=0$
$r=-1/2\pm 4i$
$y = c_1e^{-t/2}\cos 4t + c_2e^{-t/2}\sin 4t$
underdamped.

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Name. KEY

2. (ch3)
(a) [25%] The trial solution y with fewest atoms, according to the method of undetermined coefficients, contains no solution of the homogeneous equation. Explain why, using the example $y'' = 1 + x$.

(b) [75%] Determine for $y^{(4)} + y^{(2)} = x + 2e^x + 3\sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients! The trial solution should be the one with fewest atoms.

(a) The homogeneous solution $y_h = c_1 + c_2 x$ is added to y_p to obtain $y = y_h + y_p$. Therefore, y_p should be truncated to only those terms not already found in y_h.

(b) $y = x^4(x + 2e^x + 3\sin x) = x^5 + 2x^4e^x + 3x^4\sin x$

atoms = $1, x, x^2, x^3, x^4, x^5$
$e^x, xe^x, x^2e^x, x^3e^x, x^4e^x$
$\cos x, x\cos x, x^2\cos x, x^3\cos x, x^4\cos x$
$\sin x, x\sin x, x^2\sin x, x^3\sin x, x^4\sin x$

Homog. sol. contains $1, x, \cos x, \sin x$
We remove 4 from each group with the same base atom

Revised atoms = $x^2, x^3,$
$e^x,$
$x\cos x,$
$x\sin x$

trial $y = d_1x^2 + d_2x^3 + d_3e^x + d_4x\cos x + d_5x\sin x$

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Name. KEY

S2009 Midterm 2, 2280 8:35

3. (ch3 and ch7)
(a) [50%] Find by any applicable method the steady-state periodic solution for the equation $x'' + 2x' + 5x = 10\cos(t)$.
(b) [50%] Find by variation of parameters a particular solution y_p for the equation $y'' + 3y' = e^x$.

(a) $x = d_1 \cos t + d_2 \sin t$ is the unique periodic solution

$x' = -d_1 \sin t + d_2 \cos t$

$x'' = -d_1 \cos t - d_2 \sin t$

$(-d_1 \cos t - d_2 \sin t) + 2(-d_1 \sin t + d_2 \cos t) + 5(d_1 \cos t + d_2 \sin t) = 10 \cos t$

$$\begin{cases} 4d_1 + 2d_2 = 10 \\ -2d_1 + 4d_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} 4 & -2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 10 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 40 \\ 20 \end{pmatrix}$$

$x(t) = (2)\cos t + (1)\sin t$

(b) $y_h = c_1 + c_2 e^{-3x}$

$y_1 = 1$

$y_2 = e^{-3x}$

$W = \begin{vmatrix} 1 & e^{-3x} \\ 0 & -3e^{-3x} \end{vmatrix} = -3e^{-3x}$

$f = e^x$

$$y_p = \left(\int -\frac{y_2 f}{W} dx\right) y_1 + \left(\int \frac{y_1 f}{W}\right) y_2$$

$$= \left(\int \frac{-e^{-3x}}{-3e^{-3x}} e^x dx\right)\cdot 1 + \left(\int \frac{1 \cdot e^x}{-3e^{-3x}} dx\right) e^{-3x}$$

$$= \frac{1}{3}e^x + \frac{-1}{3}\frac{e^{4x}}{4} e^{-3x}$$

$$= \frac{1}{3}e^x - \frac{1}{12}e^x$$

$$= \frac{1}{4}e^x$$

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Name. KEY

S2009 Midterm 2, 2280 8:35

4. (ch7)
(a) [60%] Solve by Laplace's method $x'' + x = \cos 2t$, $x(0) = x'(0) = 0$.

(b) [15%] Assume $f(t)$ is of exponential order such that $\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}(f(t) - 1)$. Find $f(t)$.

(c) [25%] Derive an integral formula for $y(t)$ by Laplace transform methods from

$$y''(t) + y'(t) = f(t), \quad y(0) = y'(0) = 0.$$

(a) $\mathcal{L}(x'') + \mathcal{L}(x) = \mathcal{L}(\cos 2t)$

$(s^2+1)\mathcal{L}(x) = \frac{s}{s^2+4}$

$\mathcal{L}(x) = \frac{s}{(s^2+1)(s^2+4)} = \frac{a+bs}{s^2+1} + \frac{c+ds}{s^2+4}$

$= \mathcal{L}\left(a \sin t + b \cos t + c \frac{\sin 2t}{2} + d \cos 2t\right)$

$x(t) = a \sin t + b \cos t + \frac{c}{2} \sin 2t + d \cos 2t$

$a = 0, \quad b = 1, \quad c = 0, \quad d = 1$

(b) $\mathcal{L}(-t f(t)) = \mathcal{L}(f(t) - 1) \Rightarrow -t f(t) = f(t) - 1 \Rightarrow (t+1) f(t) = 1$

$f(t) = \frac{1}{1+t}$

(c) $(s^2+s)\mathcal{L}(y) = \mathcal{L}(f)$

$\mathcal{L}(y) = \frac{1}{s(s+1)} \mathcal{L}(f)$

$\frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1} = \mathcal{L}(1 - e^{-t})$

$\mathcal{L}(y) = \mathcal{L}(1 - e^{-t})\mathcal{L}(f)$

$y = \int_0^t (1 - e^{-t+x}) f(x)\,dx$

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Name. KEY

S2009 Midterm 2, 2280 8:35

5. (ch7)

(a) [40%] Solve $\mathcal{L}(f(t)) = \dfrac{1}{(s^2+s)(s^2+4s)}$ for $f(t)$.

(b) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s-1}{s^2+2s+5}$.

(c) [20%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(e^t \sin 2t)$$

(d) [20%] Solve for $f(t)$ in the relation

$$\frac{d}{ds}\mathcal{L}(f) = \left(\mathcal{L}(\cosh 4t)\right)\big|_{s\to s+3}.$$

(a) $\mathcal{L}(f) = \dfrac{1}{s^2(s+1)(s+4)} = \dfrac{a}{s} + \dfrac{b}{s^2} + \dfrac{c}{s+1} + \dfrac{d}{s+4}$

$= \mathcal{L}(a + bt + ce^{-t} + de^{-4t})$

$f(t) = a + bt + ce^{-t} + de^{-4t}$

$a = \frac{-5}{16}$, $b = \frac{1}{4}$, $c = \frac{1}{3}$, $d = \frac{-1}{48}$

use $s=\infty$ trick
$a + c + d = 0$
$a = -\frac{1}{3} + \frac{1}{48}$
$= \frac{-15}{48} = \frac{-5}{16}$

(b) $\mathcal{L}(f(t)) = \dfrac{s-1}{(s+1)^2+4} = \dfrac{s+1}{(s+1)^2+4} + \dfrac{-2}{(s+1)^2+4}$

$= \left(\dfrac{s}{s^2+4} + \dfrac{-2}{s^2+4}\right)\Big|_{s\to s+1}$

$= \mathcal{L}(\cos 2t - \sin 2t)\big|_{s\to s+1} = \mathcal{L}(e^{-t}\cos 2t - e^{-t}\sin 2t)$

$f(t) = e^{-t}\cos 2t - e^{-t}\sin 2t$

(c) $f(t) = -te^{t}\sin 2t$

(d) $-t f(t) = e^{-3t}\cosh(4t)$

$f(t) = -\dfrac{e^{-3t}\cosh(4t)}{t}$

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