Differential Equations 2280
Final Exam
Wednesday, 6 May 2009, 8:00-10:00am

Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(ch3)**

   (1a) [20%] Solve $2v'(t) = -8 + \dfrac{2}{2t+1}v(t)$, $v(0) = -4$. Show all integrating factor steps.

   (1b) [10%] Solve for the general solution: $y'' + 4y' + 6y = 0$.

   (1c) [10%] Solve for the general solution of the homogeneous constant-coefficient differential equation whose characteristic equation is $r(r^2+r)^2(r^2+9)^2 = 0$.

   (1d) [20%] Find a linear homogeneous constant coefficient differential equation of lowest order which has a particular solution $y = x + \sin\sqrt{2}x + e^{-x}\cos 3x$.

   (1e) [15%] A particular solution of the equation $mx'' + cx' + kx = F_0\cos(2t)$ happens to be $x(t) = 11\cos 2t + e^{-t}\sin\sqrt{11}t - \sqrt{11}\sin 2t$. Assume m, c, k all positive. Find the unique periodic steady-state solution x_{SS}.

   (1f) [25%] Determine for $y''' + y'' = 100 + 2e^{-x} + \sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

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2. (ch5)

(2a) [25%] A certain 2×2 matrix A has eigenpairs $\left(4, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$, $\left(6, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$. The general solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$. Eigenanalysis gives a different formula for $\mathbf{x}(t)$. Combine these facts to find e^{At}.

(2b) [25%] Let $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$. The Cayley-Hamilton method says that the solution of $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is given by $\mathbf{x}(t) = e^{3t}\mathbf{c}_1 + e^{5t}\mathbf{c}_2$. Find the constant vectors $\mathbf{c}_1$, $\mathbf{c}_2$ which produce the solution with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(2c) [20%] Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Find the general solution of $\mathbf{x}' = A\mathbf{x}$ using eigenanalysis.

(2d) [20%] Let $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. Find the general solution of $\mathbf{x}' = A\mathbf{x}$ using the exponential matrix. The Laplace resolvent formula implies $\mathcal{L}(e^{At}) = (sI - A)^{-1}$. Putzer's formula also applies. Use any method available to find e^{At}.

(2e) [10%] Let $e^{At} = \begin{pmatrix} e^t & 0 \\ te^t & e^t \end{pmatrix}$. Find A.

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3. (ch6)

(3a) [20%] Compute all equilibrium points of $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$.

(3b) [40%] For the nonlinear competition system $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$, compute the linearization at $(12, 8)$, which is a linear equation $\mathbf{u}' = A\mathbf{u}$. Classify the unique equilibrium $(0, 0)$ of $\mathbf{u}' = A\mathbf{u}$ as a node, spiral, center, saddle. What classification can be deduced for the nonlinear system?

(3c) [20%] Write the nonlinear pendulum equation $x'' + \sin x = 0$ as a nonlinear dynamical system. Then find all equilibrium points of the dynamical system.

(3d) [20%] The dynamical system $x' = y + 1$, $y' = x + y$ can be solved by various methods. Solve it and classify its unique equilibrium point $x = 1$, $y = -1$ as a node, saddle, center or spiral.

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4. (ch7)

(4a) [20%] Explain Laplace's Method, as applied to the differential equation $x'(t) + 2x(t) = e^t$, $x(0) = 1$.

(4b) [15%] Solve $\mathcal{L}(f(t)) = \dfrac{10}{(s^2+4)(s^2+9)}$ for $f(t)$.

(4c) [15%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s+3)}$.

(4d) [10%] Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}$.

(4e) [20%] Solve $x''' + x'' = 0$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = 0$ by Laplace's Method.

(4f) [20%] Solve the system $x' = x + y$, $y' = x - y + 2$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

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