Differential Equations 2280
Sample Final Exam
Wednesday, 6 May 2015, 12:45pm-3:15pm

Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for the general solution $y(x)$ in the equation $y' = 2\cot x + \dfrac{1250x^3}{1+25x^2} + x\ln(1+x^2)$.

[The required integration talent includes basic formulae, integration by parts, substitution and college algebra.]

2. (Separable Equation Test)

The problem $y' = f(x, y)$ is said to be separable provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [75%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form, but don't supply any details.

☐ $y' = -y(2xy + 1) + (2x + 3)y^2$	☐ $yy' = xy^2 + 5x^2y$
☐ $y' = e^{x+y} + e^y$	☐ $3y' + 5y = 10y^2$

(b) [25%] State a test which can verify that an equation is not separable. Apply the test to verify that $y' = x + \sqrt{|xy|}$ is not separable.

3. (Solve a Separable Equation)

Given $y^2y' = \dfrac{2x^2 + 3x}{1 + x^2}\left(\dfrac{125}{64} - y^3\right)$.

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly.

4. (Linear Equations)

(a) [60%] Solve $2v'(t) = -32 + \dfrac{2}{3t+1}v(t)$, $v(0) = -8$. Show all integrating factor steps.

(b) [30%] Solve $2\sqrt{x+2}\,\dfrac{dy}{dx} = y$. The answer contains symbol c.

(c) [10%] The problem $2\sqrt{x+2}\,y' = y - 5$ can be solved using the answer y_h from part (b) plus superposition $y = y_h + y_p$. Find y_p. Hint: If you cannot write the answer in a few seconds, then return here after finishing all problems on the exam.

5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$dx/dt = 1000\left(2 - \sqrt[5]{x}\right)^3 (2+3x)(9x^2-4)^8.$$

Expected in the diagram are equilibrium points and signs of x' (or flow direction markers $<$ and $>$).

(b) [40%] Draw a phase portrait using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

(c) [10%] Outline how to solve for non-equilibrium solutions, without doing any integrations or long details. This is a paragraph of text, with only a few equations.

6. (ch3)

(a) Solve for the general solutions:

(**a.1**) [25%] $y'' + 4y' + 4y = 0$,

(**a.2**) [25%] $y^{vi} + 4y^{iv} = 0$,

(**a.3**) [25%] Char. eq. $r(r-3)(r^3-9r)^2(r^2+4)^3 = 0$.

(b) Given $6x''(t)+7x'(t)+2x(t) = 0$, which represents a damped spring-mass system with $m = 6$, $c = 7$, $k = 2$, solve the differential equation [15%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a physical model drawing the meaning of constants m, c, k [5%].

7. (ch3)

Determine for $y^{vi} + y^{iv} = x + 2x^2 + x^3 + e^{-x} + x\sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

8. (ch3)

(a) [50%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 4x' + 6x = 10\cos(2t)$.

(b) [50%] Find by variation of parameters a particular solution y_p for the equation $y'' + 3y' + 2y = xe^{2x}$.

This instance has integration difficulties, therefore it is for a practise exam only. See Midterm 3 for an exam problem, required less integration effort.

9. (ch5)

The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix},$$

then
(a) [75%] Display eigenanalysis details for A.
(b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

10. (ch5)

(a) [20%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 4 & -2 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) [40%] Putzer's formula

$$e^{At} = r_1(t)I + r_2(t)(A - \lambda_1 I) + r_3(t)(A - \lambda_1 I)(A - \lambda_2 I)$$

uses the linear cascade

$$\begin{aligned} r_1' &= 2r_1, \quad r_1(0) = 1 \\ r_2' &= 3r_2 + r_1, \quad r_2(0) = 0 \\ r_3' &= 5r_3 + r2, \quad r_3(0) = 0. \end{aligned}$$

The general solution of $\mathbf{u}' = A\mathbf{u}$ according to Putzer's spectral formula is $\mathbf{u} = e^{At}\mathbf{u}(0)$, where e^{At} is defined above. Please compute all three coefficient functions r_1, r_2, r_3. **To save time**, don't write out e^{At} or the general solution.

(c) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton-Ziebur Method. In particular, display the equations that determine the three vectors in the general solution. **To save time**, don't solve for the three vectors in the formula.

(d) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Eigenanalysis Method. **To save time**, find one eigenpair explicitly, just to show how it is done, but don't solve for the last two eigenpairs.

(e) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to Laplace's Method. **To save time**, use symbols for partial fraction constants and leave the symbols unevaluated.

11. (ch5) Do enough to make 100%

(a) [50%] The eigenvalues are 4, 6 for the matrix $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$. Show details from either the eigenanalysis method or the Laplace method.

(b) [50%] Using the same matrix A from part (a), display the solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton Method. To save time, write out the system to be solved for the two vectors, and then stop, without solving for the vectors.

(c) [50%] Using the same matrix A from part (a), compute the exponential matrix e^{At} by any known method, for example, the formula $e^{At} = \Phi(t)\Phi^{-1}(0)$ where $\Phi(t)$ is any fundamental matrix, or via Putzer's formula.

12. **(ch5)** Do both

(a) [50%] Display the solution of $\mathbf{u}' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, using any method that applies.

(b) [50%] Display the variation of parameters formula for the system below. Then integrate to find $\mathbf{u}_p(t)$ for $\mathbf{u}' = A\mathbf{u}$.

$$\mathbf{u}' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}.$$

13. **(ch6)**
(a) Define *asymptotically stable equilibrium* for $\mathbf{u}' = \mathbf{f}(\mathbf{u})$, a 2-dimensional system.
(b) Give examples of 2-dimensional systems of type saddle, spiral, center and node.
(c) Give a 2-dimensional predator-prey example $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ and explain the meaning of the variables in the model.

14. **(ch6)**
Find the equilibrium points of $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$ and classify each linearization at an equilibrium as a node, spiral, center, saddle. What classifications can be deduced for the nonlinear system, according to the Paste Theorem?

Some maple code for checking the answers:

```
F:=unapply([14*x-x^2/2-y*x , 16*y-y^2/2 -x*y],(x,y));
Fx:=unapply(map(u->diff(u,x),F(x,y)),(x,y));
Fy:=unapply(map(u->diff(u,y),F(x,y)),(x,y));
Fx(0,0);Fy(0,0);Fx(28,0);Fy(28,0);Fx(0,32);Fy(0,32);Fx(0,32);Fy(0,32);
```

15. **(ch6)** Do enough to make 100%
(a) [25%] Which of the four types *center, spiral, node, saddle* can be unstable at $t = \infty$? Explain your answer.
(b) [25%] Give an example of a linear 2-dimensional system $\mathbf{u}' = A\mathbf{u}$ with a saddle at equilibrium point $x = y = 0$, and A is not triangular.
(c) [25%] Give an example of a nonlinear 2-dimensional predator-prey system with exactly four equilibria.
(d) [25%] Display a formula for the general solution of the equation $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{u}$.
Then explain why the system has a spiral at $(0, 0)$.
(e) [25%] Is the origin an isolated equilibrium point of the linear system $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{u}$? Explain your answer.

16. **(ch7)**

(a) Define the direct Laplace Transform.

(b) Define Heaviside's unit step function.

(c) Derive a Laplace integral formula for Heaviside's unit step function.

(d) Explain all the steps in Laplace's Method, as applied to the differential equation $x'(t) + 2x(t) = e^t$, $x(0) = 1$.

17. (ch7)

(a) Solve $\mathcal{L}(f(t)) = \dfrac{100}{(s^2+1)(s^2+4)}$ for $f(t)$.

(b) Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s-3)}$.

(c) Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}\sin(3t)$.

(d) Find $\mathcal{L}(f)$ where $f(t)$ is the periodic function of period 2 equal to $t/2$ on $0 \le t \le 2$ (sawtooth wave).

18. (ch7)

(a) Solve $y'' + 4y' + 4y = t^2$, $y(0) = y'(0) = 0$ by Laplace's Method.

(b) Solve $x''' + x'' - 6x' = 0$, $x(0) = x'(0) = 0$, $x''(0) = 1$ by Laplace's Method.

(c) Solve the system $x' = x + y$, $y' = x - y + e^t$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

19. (ch7)

(a) [25%] Solve by Laplace's method $x'' + x = \cos t$, $x(0) = x'(0) = 0$.

(b) [10%] Does there exist $f(t)$ of exponential order such that $\mathcal{L}(f(t)) = \dfrac{s}{s+1}$? Details required.

(c) [15%] Linearity $\mathcal{L}(c_1 f + c_2 g) = c_1\mathcal{L}(f) + c_2\mathcal{L}(g)$ is one Laplace rule. State four other Laplace rules. Forward and backward table entries are not rules, which means $\mathcal{L}(1) = 1/s$ doesn't count.

(d) [25%] Solve by Laplace's resolvent method

$$\begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 2x(t), \end{aligned}$$

with initial conditions $x(0) = -1$, $y(0) = 2$.

(e) [25%] Derive $y(t) = \int_0^t \sin(t-u)f(u)du$ by Laplace transform methods from the forced oscillator problem

$$y''(t) + y(t) = f(t), \quad y(0) = y'(0) = 0.$$

20. (ch7)

(a) [25%] Solve $\mathcal{L}(f(t)) = \dfrac{10}{(s^2+8)(s^2+4)}$ for $f(t)$.

(b) [25%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2(s+2)}$.

(c) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s-1}{s^2+2s+5}$.

(d) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(t^2 \sin 3t)$$

(e) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \left(\mathcal{L}\left(t^3 e^{9t} \cos 8t\right)\right)\Big|_{s \to s+3}.$$

21. (ch9)

(a) Find the Fourier sine and cosine coefficients for the 2-periodic function $f(t)$ equal to $t/2$ on $0 \le t \le 2$.

(b) State Fourier's convergence theorem.

(c) State the results for term-by-term integration and differentiation of Fourier series.

22. (ch9)

(a) Find a steady-state periodic solution by Fourier's method for $x''+x = F(t)$, where $F(t)$ is 2-periodic and equal to 10 on $0 < t < 1$, equal to -10 on $1 < t < 2$.

(b) Display Fourier's Model for the solution to the heat problem $u_t = u_{xx}$, $u(0,t) = u(1,t) = 0$, $u(x,0) = f(x)$ on $0 \le x \le 1$, $t \ge 0$.

(c) Solve $u_t = u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = 80 \sin^3 x$ on $0 \le x \le \pi$, $t \ge 0$.

23. (Vibration of a Finite String)

The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes.

Solve the finite string vibration problem on $0 \le x \le 2$, $t > 0$,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0,t) &= 0, \\ u(2,t) &= 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= -11 \sin(5\pi x). \end{aligned}$$

24. (Periodic Functions)

(a) [30%] Find the period of $f(x) = \sin(x)\cos(2x) + \sin(2x)\cos(x)$.

(b) [40%] Let $p = 5$. If $f(x)$ is the odd $2p$-periodic extension to $(-\infty, \infty)$ of the function $f_0(x) = 100x\,e^{10x}$ on $0 \le x \le p$, then find $f(11.3)$. The answer is not to be simplified or evaluated to a decimal.

(c) [30%] Mark the expressions which are periodic with letter **P**, those odd with **O** and those even with **E**.

$$\sin(\cos(2x)) \qquad \ln|2+\sin(x)| \qquad \sin(2x)\cos(x) \qquad \frac{1+\sin(x)}{2+\cos(x)}$$

25. (Fourier Series)
Let $f_0(x) = x$ on the interval $0 < x < 2$, $f_0(x) = -x$ on $-2 < x < 0$, $f_0(x) = 0$ for $x = 0$, $f_0(x) = 2$ at $x = \pm 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(a) [80%] Compute the Fourier coefficients of $f(x)$ (defined above) for the terms $\sin(67\pi x)$ and $\cos(2\pi x)$. Leave tedious integrations in integral form, but evaluate the easy ones like the integral of the square of sine or cosine.

(b) [20%] Which values of x in $|x| < 12$ might exhibit Gibb's over-shoot?

26. (Cosine and Sine Series)
Find the first nonzero term in the sine series expansion of $f(x)$, formed as the odd 2π-periodic extension of the function $\sin(x)\cos(x)$ on $0 < x < \pi$. Leave the Fourier coefficient in integral form, unevaluated, unless you can compute the value in a minute or two.

27. (Convergence of Fourier Series)

(a) [30%] Dirichlet's kernel formula can be used to evaluate the sum $\cos(2x)+\cos(4x)+\cos(6x)+\cos(8x)$. Report its value according to that formula. [Not on the final exam 2015]

(b) [40%] The Fourier Convergence Theorem for piecewise smooth functions applies to continuously differentiable functions of period p. Re-state the Fourier Convergence Theorem for the special case of a p-periodic continuously differentiable function. It is necessary to translate the results for interval $-\pi \le x \le \pi$ to the interval $-p \le x \le p$ and simplify the value to which the Fourier series converges.

(c) [30%] Give an example of a function $f(x)$ periodic of period 2 that has a Gibb's over-shoot at the integers $x = 0, \pm 2, \pm 4, \ldots$, (all $\pm 2n$) and nowhere else.