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Semester Project – Rough Draft

Due 4/1/16

The Spectral Differences in Sounds and Music

Through chemical and biological processes, we, as humans, can detect the movement and vibrations of molecules through the air and interpret them as sound. That dog barking in the distance, the light whistle of the wind through the window, the rich sound of a cello playing a long, deep note or a friend laughing in the other room. Sound is all around us. This continuous array of noise is often seen as disordered, muddled, and random. However, just as snowflakes are constructed with delicate patterns of ice crystals, sound is very formulaic and structured in its makeup. And as such, through linear algebra, we are able to measure it and compare it.

A sound spectrum is a representation of the amount of vibration at each frequency over time. Each sound has its own unique spectrum and frequency. For example, the sound of Middle C on a standard piano vibrates at 261.6 Hertz (cycles per second). A0 is 27.50 Hertz. The high pitch ringing from an old TV is about 15 kilohertz, lying just within the average adult human's range of hearing (20 Hz – 18 kHz). The more cycles per second, or rather the faster the molecular vibration that occurs as a result of the hammer hitting the piano string, the higher the pitch of the sound is. Thinner strings are able to vibrate faster than thicker ones, thus producing more vibration back and forth. The intensity, or loudness of a sound is measured in decibels (dB) and is relative to the amplitude of the sound wave. With all that said, how do we measure the spectral differences between two sounds? And what benefit would that provide us? One tool we use for sound waves in particular is Fourier Analysis.

Fourier Analysis is the study of the way complicated functions, namely Fourier series, can be represented by sums of trigonometric functions. Named after Joseph Fourier, Fourier series are wave-like functions, such as sound spectra, that can be estimated as the sum of multiple sine waves. The function is called a trigonometric polynomial and it is connected to other functions within the basis so long that for any $n \geq 1$, the set of polynomial expressions in the function is orthogonal with respect to the basis' inner product, shown in **Figure 1** below. Fourier Analysis plays a big role in understanding the spectral density, or rather the level of energy emitted by the sound.

$$\boxed{\langle f(t), g(t)\rangle = \int_0^L f(t)g(t)\,dt}$$

Figure 1: The Inner Product Formula.

To better illustrate the concept of Fourier Analysis, I have devised a small experiment. I have a friend that is a music producer, and he swears by the fact that natural instruments sound better, clearer, and purer than a computer generated sound. But I want to take that hypothesis to the next level and analyze the spectral patterns of a real instrument's note (in this experiment, we will use Middle C) and compare it with the same note of the same type of instrument, only this time it's generated by a computer application. I want to see whether the makeup and pattern of one spectrum has a unique mathematical pattern than the other and if this difference can be quantified.

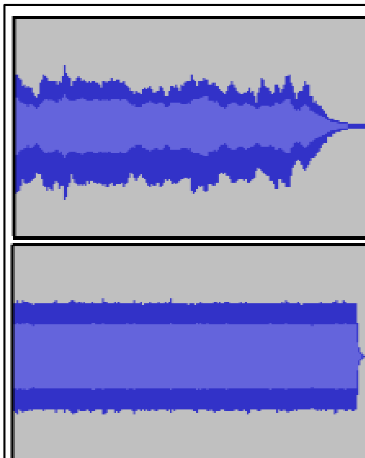

Figure 2:
Notice the difference in uniformity between the real trumpet (top) and the computer generated one (bottom)

As you can see in **Figure 2**, the spectrum of both notes are very different from each other. The real trumpet has much more variation and a disturbance in energy across the spectrum, while the computer generated sound is much more uniform and monotone. One could argue that the fluctuations in energy is due to the slight palpitations or imperfections that comes from a human blowing into an instrument. And while that may be able to partially explain the wider changes in amplitude, it would do us well to examine the waves at a much closer magnification. See **Figure 3**. The waves appear to be strikingly similar. The major difference being the tiny fluctuations in the real trumpet's sound. The computer generated trumpet follows a much more algorithmic and mathematical pattern, as is expected. But both waves are generally similar. The difference is subtle, but through this experiment we have proved that the energy spectra of each instruments' notes are indeed different not only on a grand scale, but at their very core, they are made out of different molecular vibration. This difference is what makes the real instrument sound so much more natural and, according to my friend, it contributes to a much richer and genuine sound.

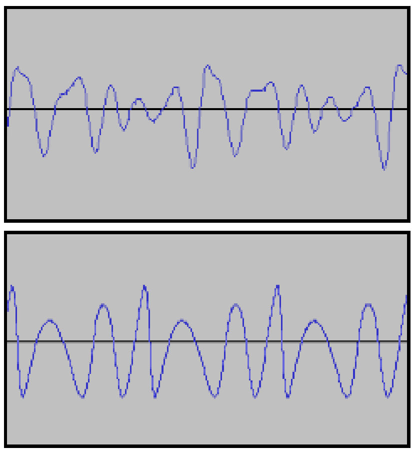

Figure 3:
Magnification of the energy spectra of the real trumpet (top) and the computer generated one (bottom)

What does this have to do with Linear Algebra? How can Linear Algebra offer us a clearer meaning? The answer lies in the application of a difference equation. As we have established before, sound is just a signal. And signals can be better analyzed and compared within their orthonormal family. Similar to the Fourier series, mentioned before, in order to find out if a signal has an orthonormal family, we must determine if it is linearly independent. In order to do that, the signal must satisfy the equation $c_1u_k + c_2v_k + c_3w_k = 0$ for all k**.** In other words, $c_1 = c_2 = c_3 = 0$. Given this, then for a set of three signals $\{u_k\}$, $\{v_k\}$, and $\{w_k\}$ we can generate the following coefficient matrix:

$$\begin{bmatrix} u_k & v_k & w_k \\ u_{k+1} & v_{k+1} & w_{k+1} \\ u_{k+2} & v_{k+2} & w_{k+2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for all } k$$

This is also known as a Casorati matrix. It was developed by Felice Casorati and is a very useful tool in studying and analyzing linear difference equations.

After analyzing and comparing energy spectra and differences between them by working through the linear systems provided by Linear Algebra, one gains a much deeper understanding of the concept and mathematics that are at work within a simple sound. Even observing the differences in amplitude and Hertz, like I have done with the real trumpet and computer generated one opens your eyes to the structure and formulaic nature of sound. They may seem chaotic and unorderly, but every sound and signal is a formula that can be analyzed and compared. And through the experiments I've performed, I have demonstrated the differences and similarities within energy spectra. With the help of Linear Algebra and modern computational tools, there can only be more progress in this fascinating area.