Quiz 4

Quiz 4, Problem 1. Picard–Lindelöf Theorem and Spring-Mass Models

Picard-Lindelöf Theorem. Let $\vec{f}(x, \vec{y})$ be defined for $|x - x_0| \le h$, $\|\vec{y} - \vec{y}_0\| \le k$, with $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ continuous. Then for some constant H, $0 < H < h$, the problem

$$\begin{cases} \vec{y}\,'(x) &= \vec{f}(x, \vec{y}(x)), \quad |x - x_0| < H, \\ \vec{y}(x_0) &= \vec{y}_0 \end{cases}$$

has a unique solution $\vec{y}(x)$ defined on the smaller interval $|x - x_0| < H$.

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The Problem. The second order problem

$$\begin{cases} u'' + 2u' + 17u = 100, \\ u(0) = 1, \\ u'(0) = -1 \end{cases} \tag{1}$$

is a spring-mass model with damping and constant external force. The variables are time x in seconds and elongation $u(x)$ in meters, measured from equilibrium. Coefficients in the equation represent mass $m = 1$ kg, a viscous damping constant $c = 2$, Hooke's constant $k = 17$ and external force $F(x) = 100$.

Convert the scalar initial value problem into a vector problem, to which Picard's vector theorem applies, by supplying details for the parts below.

(a) The conversion uses the **position-velocity substitution** $y_1 = u(x), y_2 = u'(x)$, where y_1, y_2 are the invented components of vector $\vec{y}$. Then the initial data $u(0) = 1, u'(0) = -1$ converts to the vector initial data

$$\vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) Differentiate the equations $y_1 = u(x), y_2 = u'(x)$ in order to find the scalar system of two differential equations, known as a **dynamical system**:

$$y_1' = y_2, \quad y_2' = -17y_1 - 2y_2 + 100.$$

(c) The derivative of vector function $\vec{y}(x)$ is written $\vec{y}\,'(x)$ or $\frac{d\vec{y}}{dx}(x)$. It is obtained by componentwise differentiation: $\vec{y}\,'(x) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$. The vector differential equation model of scalar system (1) is

$$\begin{cases} \vec{y}\,'(x) &= \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \\ \vec{y}(0) &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{cases} \tag{2}$$

(d) System (2) fits the hypothesis of Picard's theorem, using symbols

$$\vec{f}(x, \vec{y}) = \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \quad \vec{y}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The components of vector function $\vec{f}$ are continuously differentiable in variables x, y_1, y_2, therefore $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ are continuous.

Quiz4 Problem 2. The velocity of a crossbow bolt launched upward from the ground was determined from a video and a speed gun to complete the following table.

Time t in seconds	Velocity $v(t)$ in ft/sec	Location
0.000	60	Ground
1.7	0	Maximum
3.5	-52	Near Ground Impact

(a) The bolt velocity can be approximated by a quadratic polynomial

$$v(t) = at^2 + bt + c$$

which reproduces the table data. Find three equations for the coefficients a, b, c. Then solve for the coefficients.

(b) Assume a linear drag model $v' = -32 - \rho v$. Substitute the polynomial answer of (a) into this differential equation, then substitute $t = 0$ and solve for $\rho \approx 0.11$.

(c) Solve the model $w' = -32 - \rho w$, $w(0) = 60$ with $\rho = 0.11$.

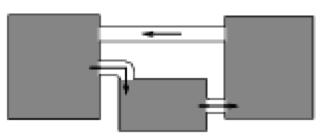
(d) The error between $v(t)$ and $w(t)$ can be measured. Is the drag coefficient value $\rho = 0.11$ reasonable?

References. Edwards-Penney sections 2.3, 3.1, 3.2. Course documents on Linear algebraic equations and Newton kinematics.

Quiz4 Extra Credit Problem 3. Consider the system of differential equations

$$\begin{aligned} x_1' &= -\tfrac{1}{5}x_1 && && + \tfrac{1}{7}x_3, \\ x_2' &= \tfrac{1}{5}x_1 && - \tfrac{1}{3}x_2, && \\ x_3' &= && \tfrac{1}{3}x_2 && - \tfrac{1}{7}x_3, \end{aligned}$$

for the amounts x_1, x_2, x_3 of salt in recirculating brine tanks, as in the figure:



Recirculating Brine Tanks A, B, C

The volumes are $50, 30, 70$ for A, B, C, respectively.

The steady-state salt amounts in the three tanks are found by formally setting $x_1' = x_2' = x_3' = 0$ and then solving for the symbols x_1, x_2, x_3.

(a) Solve the corresponding linear system of algebraic equations for answers x_1, x_2, x_3.

(b) The total amount of salt is uniformly distributed in the tanks in ratio 5 : 3 : 7. Explain this mathematically from the answer in (a).

References. Edwards-Penney sections 3.1, 3.2, 7.3 Figure 5. Course documents on Linear algebraic equations and Systems and Brine Tanks.