Name Student Solutions 6May2015

Math 2250-10 Final Exam for 7:15am on 6 May 2015

Scores	
Ch1-2.	100
Ch3.	100
Ch4.	100
Ch5.	100
Ch6.	100
Ch7.	100
Ch9.	100
Ch10.	100

A A A

Ch1 and Ch2. (First Order Differential Equations)

A [20%] **Ch1-Ch2(a):**
Find the position $x(t)$ from the velocity model $\frac{d}{dt}\left(e^t v(t)\right) = 10e^{2t}$, $v(0) = 0$ and the position model $\frac{dx}{dt} = v(t)$, $x(0) = 100$.

A [10%] **Ch1-Ch2(b):**
Find all equilibrium solutions for $y' = x^3 e^y(2 + \cos(y))(y^2 - 3y + 2)$.

[20%] **Ch1-Ch2(c):**
A Given $y' = \frac{2x^2 + x}{1 + x}\left(\frac{y^4 - 2y^2 + 1}{y}\right)$,
find the non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly.

[20%] **Ch1-Ch2(d):**
A Solve the linear homogeneous equation $2\sqrt{1+x}\frac{dy}{dx} = y$ using the integrating factor shortcut.

[10%] **Ch1-Ch2(e):**
Draw a phase line diagram for the differential equation

A
$$\frac{dx}{dt} = (3 + x)(x^2 - 9)(4 - x^2)^3.$$

Label the equilibrium points, display the signs of dx/dt, and classify each equilibrium point as funnel, spout or node. To save time, **do not draw** a phase portrait.

A [20%] **Ch1-Ch2(f):**
Solve the linear drag model $1000\frac{dv}{dt} = 50 - 200v$ using superposition $v = v_h + v_p$.

a) $\int \frac{d}{dt}\left(e^t v(t)\right) = \int 10e^{2t}\,dt$

$e^t \cdot v(t) = 5e^{2t} + C$

$v(t) = 5e^t + Ce^{-t}$, $v(0) = 0 = 5 + C$, $C = -5$

$v(t) = 5e^t - 5e^{-t}$

$v(t) = \frac{dx}{dt}$, $x(t) = \int 5e^t - 5e^{-t}$

$x(t) = 5e^t + 5e^{-t} + C$, $x(0) = 100 = 5 + 5 + C$, $C = 90$

$x(t) = 5e^t + 5e^{-t} + 90$

b) $\frac{dy}{dx} = x^3 \cdot e^y(2 + \cos(y))(y^2 - 3y + 2)$

equil. solns are for all $G(y) = 0$

$y^2 - 3y + 2 = 0$ $(y-1)(y-2) = 0$ solns: $y = 1, 2$

c) $\dfrac{dy}{dx} = \left[\dfrac{2x^2+x}{1+x}\right]\left[\dfrac{y^4-2y^2+1}{y}\right]$ by separation

$$\int \frac{y}{y^4-2y^2+1}\,dy = \int \frac{2x^2+x}{1+x}\,dx$$

$$\int \frac{y}{(y^2-1)(y^2-1)}\,dy = \int \frac{2x^2+x}{1+x}\,dx \qquad u = 1+x \qquad \int \frac{2(u-1)(u)}{u}$$

$$\frac{1}{2-2y^2} = x(x+1) + \ln|x+1| + C$$

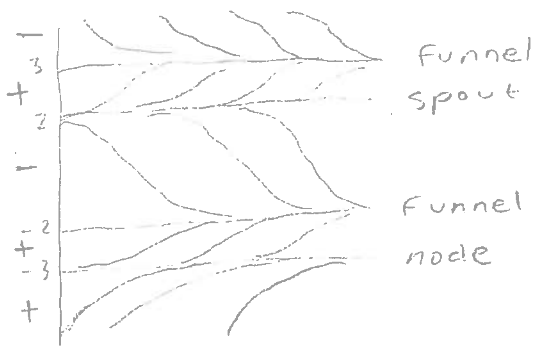
d) $(2\sqrt{1+x})\cdot y' - y = 0$

$$y' - \left[\frac{1}{2\sqrt{1+x}}\right] y = 0$$

$$p = e^{\int \frac{1}{2\sqrt{1+x}}dx} = e^{\sqrt{x+1}} \qquad u = x+1$$

$$e^{\sqrt{x+1}}\cdot y = \int 0\cdot e^{\sqrt{x+1}} \qquad y = \frac{C}{e^{\sqrt{x+1}}}$$

e) $\dfrac{dx}{dt} = (3+x)(x^2-9)(4-x^2)^3$ solns: $-3, -2, 2, 3$



F) $v_h \Rightarrow 1000v' + 200v = 0$, $v' + 0.2v = 0$ $\qquad p = e^{\int 0.2\,dt} = e^{0.2t}$

$$v_h = \frac{C}{e^{0.2t}} = Ce^{-0.2t}$$

$$v_p = 0.25 \qquad v = v_h + v_p = 0.25 + Ce^{-0.2t}$$

Name ___ ___ 6May2015

Math 2250-10 Final Exam for 7:15am on 6 May 2015

100

Ch3. (Linear Systems and Matrices)

[40%] Ch3(a): Consider a 3×5 matrix A and its reduced row echelon form:

A

$$B = \text{rref}(A) = \begin{pmatrix} \boxed{1} & 2 & 0 & 2 & 4 \\ 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(1) Explain in detail why $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have exactly the same solutions.

(2) Show the linear algebra steps used to find the scalar general solution to the system $B\vec{x} = \vec{0}$.

(3) Report a basis for the solution space S of $A\vec{x} = \vec{0}$.

(4) Report the dimension of S.

[20%] Ch3(b): Define matrix A and vector $\vec{b}$ by the equations

A

$$A = \begin{pmatrix} -2 & 3 \\ 0 & 4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_1, x_2 by Cramer's Rule, showing **all details** (details count 75%).

[40%] Ch3(c):

A

Determine which values of k correspond to (1) a unique solution, (2) infinitely many solutions and (3) no solution, for the system $A\mathbf{x} = \mathbf{b}$ given by

$$A = \begin{pmatrix} 0 & k-2 & k-3 \\ 1 & 4 & k \\ 1 & 4 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ k \end{pmatrix}.$$

a1) both augmented matrices A + B are augmented with $\vec{0}$ so solving for $\vec{x}$ will lead to the same set of solutions. It's homogeneous

a2) From B, $x_1 + 2x_2 + 2x_4 + 4x_5 = 0$

$x_3 + x_4 + x_5 = 0$

$0 = 0$

free vars = x_2, x_4, x_5, let $x_2 = t_1$ $x_4 = t_2$ $x_5 = t_3$

so $\vec{x} = \begin{bmatrix} -2t_1 - 2t_2 - 4t_3 \\ t_1 \\ -t_2 - t_3 \\ t_2 \\ t_3 \end{bmatrix}$

a3) for t_1, t_2, t_3

partial's $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

a4) dim(s) = # of basis vectors

= 3

b) $A = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$ $b = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ $A_{x_1} = \begin{bmatrix} -3 & 3 \\ 1 & 4 \end{bmatrix}$

$A_{x_2} = \begin{bmatrix} -2 & -3 \\ 0 & 1 \end{bmatrix}$

$|A| = (-2)(4) - (3)(0) = -8$

$|A_{x_1}| = (-3)(4) - (3)(1) = -15$

$|A_{x_2}| = (-2)(1) - (-3)(0) = -2$

$X_1 = \dfrac{|A_{x_1}|}{|A|} = \dfrac{15}{8}$ $X_2 = \dfrac{|A_{x_2}|}{|A|} = \dfrac{2}{8} = \dfrac{1}{4}$

c) $\begin{bmatrix} 0 & k-2 & k-3 \\ 1 & 4 & k \\ 1 & 4 & 3 \end{bmatrix}$ $\det(A) = -(k-2)(3-k) + (k-3)(0)$

so there's unique solns for all $k \neq 2, 3$

for $k = 2$

$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{bmatrix}$ $k_2 =$ no soln, signal equation

$k = 3$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $k = 3$ infinite solns

X_3 free variable $0 = 0$

Name _

6May2015

Math 2250-10 Final Exam for 7:15am on 6 May 2015

Ch4. (Vector Spaces)

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A

[30%] **Ch4(a)**: Some independence tests below apply to prove that vectors x, x^2, xe^x are independent in the vector space of all continuous functions on $-\infty < x < \infty$. Mark one method and display the details of application (details count 75%).

☐	**Wronskian test**	Wronskian of functions f, g, h nonzero at $x = x_0$ implies independence of f, g, h.
☒	**Atom test**	Any finite set of distinct Euler solution atoms is independent.
☐	**Sampling test**	Let samples a, b, c be given and for functions f, g, h define $$A = \begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(c) & g(c) & h(c) \end{pmatrix}.$$ Then $\det(A) \neq 0$ implies independence of f, g, h.

A

[20%] **Ch4(b)**: Give an example of three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ for which the nullity of their augmented matrix A is two.

A

[20%] **Ch4(c)**: Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fraction decomposition of the fraction

$$\frac{3x^2 - 14x + 3}{(x+1)^2(x-2)^2}$$

To save time, **do not solve** for a, b, c, d.

A

[30%] **Ch4(d)**:

The 5×6 matrix A below has some independent columns. Report a largest set of independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & -2 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 6 & 0 & 0 & 3 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}$$

a) x, x^2, xe^x are all Euler atoms for roots 0,0,0,1,1 so the functions are independent

b) $A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ only has 1 pivot col. nullity = 2 $\quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

c) $\dfrac{3x^2-14x+3}{(x+1)^2(x-2)^2} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x-2)} + \dfrac{D}{(x-2)^2}$

$3x^2-14x+3 = A(x+1)(x-2)^2 + B(x-2)^2 + C(x+1)^2(x-2) + D(x+1)^2$

$x^3-4x^2+4x+x^2-4x+4$ x^2-4x+4 $x^3:\ -3x-2$ x^2+2x+1

x^3-3x^2+4

$$\begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 1 & -4 & 4 \\ 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -14 \\ 3 \end{bmatrix}$$

d)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -3 & 0 & -2 & 0 & 1 & -1 \\ 6 & 0 & 6 & 0 & 0 & 3 \\ 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

find rref

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & -2 & -1 \\ 6 & 0 & 6 & 0 & 0 & 3 \\ 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot cols. are v_1 & v_3 so they are the largest set of ind vectors

Name ___ ______ 6May2015

Math 2250-10 Final Exam for 7:15am on 6 May 2015

Ch5. (Linear Equations of Higher Order) 100

A [10%] **Ch5(a)**: Find a basis for the solution space of $y'' + 4y' + 5y = 0$.

A [20%] **Ch5(b)**: Solve for the general solution y of the equation $\frac{d^6y}{dx^6} + 16\frac{d^4y}{dx^4} = 0$.

A [20%] **Ch5(c)**: Find a basis for the solution space of a linear constant coefficient homogeneous differerential equaiton, given the characteristic equation is $r(r+1)(r^3-r)^2(r^2+2r+5)^2 = 0$.

A [20%] **Ch5(d)**: Given $6x''(t) + 2x'(t) + 2x(t) = 5\cos(\omega t)$, which represents a damped forced spring-mass system with $m = 6$, $c = 2$, $k = 2$, answer the following questions.

True [ ] or False [X]. Practical mechanical resonance is at input frequency $\omega = \sqrt{5/2}$.

True [ ] or False [X]. The homogeneous problem is over-damped.

A [30%] **Ch5(e)**: Determine for $\frac{d^5y}{dx^5} + 4\frac{d^3y}{dx^3} = x + x^2 + e^x + x\cos(2x)$ the shortest trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

a) $r^2 + 4r + 5 =$ roots at $-2 \pm i$

which has euler atoms $e^{-2t}\cos(t)$, $e^{-2t}\sin(t)$

so basis would be any linear comb. of these atoms.

b) $y^{(6)} + 16y^{(4)} = 0 = r^6 + 16r^4 = r^4(r^2+16)$

roots: $0,0,0,0, \pm 4i$

atoms: $1, x, x^2, x^3, \cos(4t), \sin(4t)$

$y = C_1 + C_2x + C_3x^2 + C_4x^3 + C_5\cos(4t) + C_6\sin(4t)$

c) $r(r+1)[r(r^2-1)]^2(r^2+2r+5)^2 = 0$

roots: $0,0,0,-1,-1,-1,1,1,-1\pm 2i, -1\pm 2i$

atoms: $1, x, x^2, e^{-x}, xe^{-x}, x^2e^{-x}, e^x, xe^x, e^{-x}\cos(2x), e^{-x}\sin(2x)$
$xe^{-x}\cos(2x), xe^{-x}\sin(2x)$

basis would be any linear comb of this group of atoms

d) for RLC $C = \frac{1}{m}$ $L = k$ $R = c$ $\omega = \sqrt{LC} = \sqrt{\frac{k}{m}} \neq \sqrt{5/2}$

$6r^2 + 2r + 2$ roots: $\frac{-2 \pm \sqrt{4-48}}{12}$ underdamped

e) $y^{(5)} + 4y^{(3)} = x + x^2 + e^x + x\cos(2x)$

$r^3(r^2+4)=0$ (homogeneous)

LHS

atoms: $1, x, x^2$

$\cos(2x)$

$\sin(2x)$

RHS

atoms 1) $1, x, x^2$

2) e^x

3) $\cos(2x)$, $x\cos(2x)$

4) $\sin(2x)$, $x\cos(2x)$

Conflict with groups 1, 3, 4

new group = x^3, x^4, x^5

e^x

$x\cos(2x)$, $x^2\cos(2x)$

$x\sin(2x)$, $x^2\sin(2x)$

shortest trial soln is any linear comb. of new group.

Name ______________ 6May2015

Math 2250-10 Final Exam for 7:15am on 6 May 2015

100

Ch6. (Eigenvalues and Eigenvectors)

A [20%] **Ch6(a):** Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\boxed{\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)}, \quad \cancel{\left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)}, \quad \boxed{\left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)}.$$

A [20%] **Ch6(b):** Find the 2×2 matrix A which has eigenpairs

$$\left(0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

A [20%] **Ch6(c):** Find the eigenvectors corresponding to complex eigenvalues $-1 \pm 3i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$.

A [30%] **Ch6(d):** Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}$. Display the details for finding all eigenpairs of A.

a) $A\vec{v} = \lambda\vec{v}$ | $\begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} -2 \\ -3 \end{pmatrix}$?

$\begin{pmatrix} -7+8 \\ -12+14 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ✓ | $\begin{pmatrix} -14+4 \\ -24+7 \end{pmatrix} = \begin{pmatrix} -10 \\ -17 \end{pmatrix} \neq \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} -14+12 \\ -24+21 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$\begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ ✓

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b) $AP = PD$; $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$; $P = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$A = PDP^{-1}$

$P^{-1} = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = P^{-1}$

$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1(0)+2(0) & 1(0)+(2)(2) \\ 0(0)+1(0) & (0)(0)+(2)(1) \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1(0)+4(0) & 0(-2)+4 \\ 0(1)+2(0) & (0)(-2)+2 \end{pmatrix}$

$= \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}$

$\boxed{A = \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}}$

Check: $\begin{vmatrix} -\lambda & 4 \\ 0 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda) = 0$

$-2\lambda + \lambda^2 = 0$

$\boxed{\lambda = 0 ; \lambda = 2}$ ✓

$\begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}$

$\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ ✓

6c] $A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$ $\lambda = -1 \pm 3i$

$(A - \lambda I)\vec{v} = \vec{0}$

$\lambda = -1 + 3i$

$\begin{pmatrix} -1-(-1+3i) & 3 \\ -3 & -1-(-1+3i) \end{pmatrix} = \begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix} \xrightarrow{\text{mult}(1/3)} \begin{pmatrix} -1i & 1 \\ -1 & -1i \end{pmatrix} \xrightarrow{\text{mult}(R_1, i)} \begin{pmatrix} -1i^2 & 1i \\ -1 & -1i \end{pmatrix}$ $\to i^2 = -1$

$\begin{pmatrix} 1 & 1i \\ -1 & -1i \end{pmatrix} \to \begin{pmatrix} 1 & 1i \\ 0 & 0 \end{pmatrix}$ RREF

$x_1 = -it_1$
$x_2 = t_1$ $\Rightarrow \begin{pmatrix} -i \\ 1 \end{pmatrix}$ and

so conjugate is: $\begin{pmatrix} i \\ 1 \end{pmatrix}$
(swap signs on i)

$\boxed{\text{Pairs: } \left(-1+3i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right) ; \left(-1-3i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)}$

6d] $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}$

$\to$ Pairs are: $\left(-5, \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}\right)$ $\left(-1, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right)$ $\left(1, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)$

1) $|A - \lambda I| = 0$ $= \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 0 & 0 & -5-\lambda \end{vmatrix} = (-5-\lambda) \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$
$= (-5-\lambda)(\lambda^2 - 1) = 0$
$\lambda = 5, \pm 1$

$\lambda_1 = -5, \ \lambda_2 = 1, \ \lambda_3 = -1$

2) $(A - \lambda I)\vec{v} = \vec{0}$

$\lambda_1 = -5$

$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & -24 & -4 \\ 1 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 6 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 1 \\ 0 & 1 & 1/6 \\ 0 & 0 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 1/6 \\ 0 & 1 & 1/6 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ $x_1 = -1/6\, t_1$, $x_2 = -1/6\, t_1$, $x_3 = t_1$ $\Rightarrow \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$

$\lambda_2 = -1$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 = -t_1$
$x_2 = t_1$ $\Rightarrow \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
$x_3 = 0$

$\lambda_2 = 1$

$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -6 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 = 1t_1$
$x_2 = t_1$
$x_3 = 0$ $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Name 6May2015

Math 2250-10 Final Exam for 7:15am on 6 May 2015

100

Ch7. (Linear Systems of Differential Equations)

The methods studied are (1) Linear Cascade method, (2) Cayley-Hamilton-Ziebur method and the 2×2 shortcut, (3) Eigenanalysis method, (4) Laplace's method, including the Laplace resolvent shortcut.

A [30%] **Ch7(a)**: Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\frac{dx}{dt} = x + y,$$
$$\frac{dy}{dt} = 6x + 2y.$$

A [30%] **Ch7(b)**: Consider the scalar system

$$\begin{cases} x' = 3x \\ y' = x, \\ z' = x + y \end{cases}$$

Report two possible methods that apply to solve for x, y, z. Then choose one method and display the solution details and the answer (details count 75%). (1) Lac

A [40%] **Ch7(c)**: Define

$$A = \begin{pmatrix} -3 & 4 & -10 \\ 0 & 2 & 0 \\ 5 & -4 & 12 \end{pmatrix}$$

The eigenvalues of A are 7, 2, 2. Apply the eigenanalysis method, which requires eigenvalues and eigenvectors, to solve the differential system $\mathbf{u}' = A\mathbf{u}$.

a) $x' = x + y$
$y' = 6x + 2y$

$\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$; $A = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$

$u' = A\vec{u}$; $|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 1 \\ 6 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(2-\lambda) - 6 = 0$

$2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$

$\lambda_1 = 4$, $\lambda_2 = -1$

$(A - \lambda I)\vec{v} = \vec{0}$

$\lambda_1 = 4$: $\begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix} \to \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1/3 \\ 0 & 0 \end{pmatrix}$

$x_1 = 1/3\, t_1$, $x_2 = t_1$ $\to \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\lambda_2 = -1$: $\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix}$

$x_1 = -1/2\, t_1$, $x_2 = t_1$ $\to \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ← whew it was diagonalizable

Eigenpairs: $\left(4, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)$; $\left(-1, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right)$

$$\boxed{\vec{u} = c_1 e^{4t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

7b

$x' = 3x$
$y' = x$
$z' = x + y$

$\rightarrow \begin{pmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

① triangular, so use linear cascades

② Could also use Laplace's method (always works)

$x' = 3x$

$x - 3x = 0$

$x = \frac{c_1}{w}$; $w = e^{-3\int dt} = e^{-3t}$

$= \frac{c_1}{e^{-3t}}$

$x = c_1 e^{3t}$

$y' = x$

$\int y' dt = \int c_1 e^{3t} dt$

$y = \frac{c_1 e^{3t}}{3} + c_2$

$z' = x + y$

$= c_1 e^{3t} + \frac{c_1 e^{3t}}{3} + c_2$

$\int z' dt = \int \left(\frac{4}{3} c_1 e^{3t} + c_2\right) dt$

$= \frac{4}{3} \frac{c_1 e^{3t}}{3} + c_2 t + c_3$

$z = \frac{4}{9} c_1 e^{3t} + c_2 t + c_3$

$$\boxed{\begin{aligned} x &= c_1 e^{3t} \\ y &= \frac{1}{3} c_1 e^{3t} + c_2 \\ z &= \frac{4}{9} c_1 e^{3t} + c_2 t + c_3 \end{aligned}}$$

7c

Eigenvalues: 7, 2, 2 $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\boxed{u = c_1 e^{7t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}$$

$\lambda_1 = 7$

$\begin{pmatrix} -10 & 4 & -10 \\ 0 & -5 & 0 \\ 5 & -4 & 5 \end{pmatrix}$

$\begin{pmatrix} -10 & 4 & -10 \\ 0 & -5 & 0 \\ 0 & -2 & 0 \end{pmatrix}$

$\begin{pmatrix} -10 & 0 & -10 \\ 0 & -5 & 0 \\ 0 & -2 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 = -t_1$
$x_2 = 0$
$x_3 = t_1$
$\Rightarrow \underset{\vec{v}_1}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

$\lambda_2 = 2$

$\begin{pmatrix} -5 & 4 & -10 \\ 0 & 0 & 0 \\ 5 & -4 & 10 \end{pmatrix}$

$\begin{pmatrix} -5 & 4 & -10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & -4/5 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 = 4/5 t_1 - 2t_2$
$x_2 = t_1$
$x_3 = t_2$

$\frac{\partial \vec{x}}{\partial t_1} = \begin{pmatrix} 4/5 \\ 1 \\ 0 \end{pmatrix}$ $\quad \frac{\partial \vec{x}}{\partial t_2} = \underset{\vec{v}}{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}$

$\underset{\vec{v}_2}{\begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}}$

Name ______________________ 6May2015

Math 2250-10 Final Exam for 7:15am on 6 May 2015

Ch9. (Nonlinear Systems) 100

A [10%] **Ch9(a)**: Which of the four types *center*, *spiral*, *node*, *saddle* can be asymptotically stable at $t = \infty$? Explain your answer.

In parts (b), (c), (d), (e) below, consider the **nonlinear system**

$$x' = 14x - 2x^2 - xy, \quad y' = 16y - 2y^2 - xy. \tag{1}$$

A [20%] **Ch9(b)**: Display details for finding the equilibrium points for the nonlinear system (1). There are four answers, one of which is (4, 6).

A [20%] **Ch9(c)**: Consider again system (1). Compute the Jacobian matrix at (x, y). Then compute the Jacobian matrix at equilibrium point (4, 6).

A [20%] **Ch9(d)**: Classify the linear system for equilibrium (4, 6) as a node, spiral, center, saddle.

A [30%] **Ch9(e)**: Consider again system (1). What classification can be deduced for equilibrium (4, 6) of nonlinear system (1), according to the Pasting Theorem? Explain fully (details count 75%).

a) Spiral or node : ~~center~~ = always stable (so not asymptotically!)
Spiral : stable if (0,0) is approached in real (positive) time
node : stable if (0,0) " " " ")

b) $x' = 14x - 2x^2 - xy$

$0 = 14x - 2x^2 - xy$

$0 = x(14 - x - y)$

$0 = 0$

1) $0 = x$

$y = 0$: $0 = x(14 - x - 0)$

1) $0 = x$

3) or $x = 14$

$0 = 14 - x - y$

$y' = 16y - 2y^2 - xy$

$0 = (16 - 2y - x)y$

1, $x = 0$: $0 = (16 - 2y - 0)y$

$0 = (16 - 2y)y$

1) $0 = y$

2) or $y = 8$

$y = 0$: $0 = (16 - 2y - x)(0)$

$0 = 0$ ✓

$0 = 16 - 2y - x$

$x = 16 - 2y$

1) (0, 0)
2) (0, 8)
3) (14, 0)
4) (4, 6) ← given

9c] $J(x_0, y_0) = \left\langle \frac{\partial \vec{f}}{\partial x} \middle| \frac{\partial \vec{f}}{\partial y} \right\rangle$

$f_1 = 14x - 2x^2 - xy$ $\qquad$ $f_2 = 16y - 2y^2 - xy$

$\frac{\partial f_1}{\partial x} = 14 - 4x - y$ $\qquad$ $\frac{\partial f_2}{\partial x} = 0 - 0 - y = -y$

$\frac{\partial f_1}{\partial y} = 0 - 0 - x = -x$ $\qquad$ $\frac{\partial f_2}{\partial y} = 16 - 4y - x$

$$J(x, y) = \begin{pmatrix} 14-4x-y & -x \\ -y & 16-4y-x \end{pmatrix}$$

$$J(4,6) = \begin{pmatrix} 14-4(4)-(6) & -4 \\ -6 & 16-(4)(6)-4 \end{pmatrix}$$

$$= \begin{pmatrix} 14-16-6 & -4 \\ -6 & 16-24-4 \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} -8 & -4 \\ -6 & -12 \end{pmatrix}}$$

-2

d] $|J - \lambda I| = 0$

$$\begin{vmatrix} -8-\lambda & -4 \\ -6 & -12-\lambda \end{vmatrix} = 0$$

$(-8-\lambda)(-12-\lambda) - 24$

$= 96 + 8\lambda + 12\lambda + \lambda^2 - 24 = 0$

$= \lambda^2 + 20\lambda + 72 = 0$

$\lambda = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$= \frac{-20}{2} \pm \frac{\boxed{\sqrt{20^2 - 4(72)}}}{2}$

$= -10 \pm \frac{\boxed{\sqrt{184}}}{2}$ → 13 or 14 or 15

← so these are different, negative (always) values

96
$\times 24$
$= 72$

$72 = 7 \times 8$
$= 12 \cdot 6$

$20 \times 20 = 400$ $\qquad$ $72 \times 4 = 288$ $\qquad$ $400 - 216 = 184$

e^{-at}, e^{-bt}

→

So given there's no sin, cos;

- NO Rotation (~~center~~, ~~spiral~~ = out)
- e^{-at}, e^{-bt} limits to 0,0 as $t \to \infty$
  - b/c it approaches a fixed point, throw out ~~saddle~~
  - b/c it limits as $t \to \infty$, we know it's stable

$\boxed{\text{Stable NODE}}$

e) the only exceptions to pasting theorem (which says the phase portrait picture for linear system persists locally) are:

1) center: $\to$ center, spiral (stable), spiral unstable
2) node w/ = eigenvalues $\Rightarrow$ spiral or node w/ same stability

- because we have different eigen values, there is no exception, and the same classification holds ($\boxed{\text{Stable, node}}$)

Name ______________________________ 6May2015

Math 2250-10 Final Exam for 7:15am on 6 May 2015

Ch10. (Laplace Transform Methods)

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

100

[40%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables. Each wrong answer subtracts 3 points from the total of 40.

$\frac{2}{3}\left(\frac{1}{s+\frac{1}{3}}\right)$ $\frac{5}{2}\left(\frac{2}{s^2+4}\right)$ $\frac{1}{2}\left(\frac{2}{(s+1)^2+4}\right)$

$f(t)$	$2\cos(t)$	$\frac{2}{3}e^{-\frac{1}{3}t}$	$\frac{5}{2}\sin 2t$	$2t+1$	$\frac{1}{2}e^{-t}\sin 2t$
$\mathcal{L}(f(t))$	$\frac{2s}{1+s^2}$	$\frac{2}{3s+1}$	$\frac{5}{s^2+4}$	$\frac{2}{s^2}+\frac{1}{s}$	$\frac{1}{(s+1)^2+4}$

$2\left(\frac{s}{s^2+1}\right)$

$f(t)$	t	$(1+e^{-t})^2$	$\cos(t)$	$e^{-t}\cos(t)$	$t\sin t$
$\mathcal{L}(f(t))$	$\frac{1}{s^2}$	$1+2e^{-t}+e^{-2t}$ $\frac{1}{s}+\frac{2}{s+1}+\frac{1}{s+2}$	$\frac{s}{s^2+1}$	$\frac{s+1}{(s+1)^2+1}$	$\frac{-2s}{(s^2+1)^2}$

$\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$

$\frac{d}{ds}(s^2+1)^{-1}$

$\frac{-2s}{(s^2+1)^2}$

A [30%] **Ch10(b)**: Let u be the unit step, $u(t)=1$ for $t \ge 0$, $u(t)=0$ for $t<0$. Compute $\mathcal{L}(x(t))$, given the mechanical problem

$$x''(t)+4x(t)=5(u(t-1)-u(t-2)), \quad x(0)=x'(0)=0.$$

To save time, **do not solve** for $x(t)$.

A [30%] **Ch10(c)**: Solve for $g(t)$ in the equation $\mathcal{L}(g(t))=\frac{e^{-2s}}{s+5}$.

C $e^{-2s}\left(\frac{1}{s+5}\right)$

$\left(\frac{1}{s+5}\right)\Big|_{t-2}$

$\frac{1}{s+5}=e^{-5t}$

B

$-x'(0)+s[-x(0)+sX(s)]+4X(s)=5\frac{e^{-s}}{s}-5\frac{e^{-2s}}{s}$

$(s^2+4)X(s)=\frac{5}{s}(e^{-s}-e^{-2s})$

$$\boxed{X(s)=\frac{5(e^{-s}-e^{-2s})}{s(s^2+4)}}$$

C

$$\Rightarrow \boxed{g(t)=e^{-5(t-2)}u(t-2)}$$