Name ________________________

Math 3150 Problems
Weeks 11-13
June 2009

Due date: See the internet due date. Problems are collected twice a week. Records are locked when the stack is returned. Records are only corrected, never appended.

Submitted work. Please submit one stapled package per problem. Label each problem with its corresponding problem number, e.g., Prob3.1-4 or Xc1.2-4. Kindly label extra credit problems with label **Extra Credit**. You may attach this printed sheet to simplify your work.

Labeling. The label Probx.y-z means the problem is for chapter x, section y, problem z. When $y = 0$, then the problem does not have a textbook analog, it is a **background problem**. Otherwise, the problem number should match a corresponding problem in the textbook. The same labeling applies to extra credit problems, e.g., Xc1.0-4, Xc1.1-2.

Week 11: 4.1-4.2 – Laplacian and Symmetric Drumhead Vibration

Prob4.1-5. (Laplacian in Spherical Coordinates)

Represent $u(x,y,z) = \left(x^2+y^2+z^2\right)^{3/2}$ in spherical coordinates $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$ and then decide if u satisfies Laplace's equation in spherical coordinates,

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + \cot\theta\, u_\theta + \csc^2\theta\, u_{\phi\phi}\right) = 0.$$

Xc4.1-9. (Spherical Laplacian Symmetric Case)

Supply details when $u(r,\theta,\phi)$ is independent of θ and ϕ to verify that Laplace's equation

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + \cot\theta\, u_\theta + \csc^2\theta\, u_{\phi\phi}\right) = 0$$

reduces to the simpler symmetric case equation

$$u_{rr} + \frac{2}{r}u_r = 0.$$

Prob4.0-1. (Power Series Method)

Solve $y'' + y = x + 1$, $y(0) = 0$, $y'(0) = 1$ by the power series method to obtain a series solution in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Answer: $y(x) = x + 1 - \cos x = x + 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$. To justify your answer, which may have a different form, compare the first five nonzero terms of the two series answers.

Prob4.0-2. (Euler Differential Equation)

The transformation pair $x = e^t$, $y(x) = u(t)$ changes the Euler differential equation $Ax^2y'' + Bxy' + Cy = 0$ into the constant-coefficient equation

$$A\left(\frac{d^2u}{dt^2} - \frac{du}{dt}\right) + B\frac{du}{dt} + Cu = 0.$$

(a) Solve $x^2y'' + 4xy' + 2y = 0$.

(b) Solve $2x^2y'' + 6xy' + 2y = 0$.

(c) Solve $x^2y'' + xy' + 4y = 0$.

(d) Solve $x^2y'' - xy' + 5y = 0$.

Prob4.0-3. **(Frobenius Method)**

Solve by the Frobenius method for $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, $x > 0$, where r is the largest root of the indicial equation. Find only the first three nonzero terms of the Frobenius series. Check the answer using a computer algebra system.

(a) $4xy'' + 6y' + y = 0$

(b) $4xy'' + 2y' + y = 0$

(c) $2y'' - \frac{1}{x}y' + \frac{2}{x}y = 0$

Prob4.0-4. **(Frobenius Method Case 3)**

Solve for two independent solutions of $xy'' - (2+x)y' + 2y = 0$ using the method of Frobenius and optionally a computer algebra system.

References: Asmar *PDE and BVP*, Appendix A.5 or A.6, and Edwards-Penney *DE and BVP*, section 8.4.

Prob4.2-1. **(Radially Symmetric Drumhead)**

Solve the radially symmetric drumhead problem for $u(r,t)$ on the domain $0 < r < 2$, $t > 0$:

$$\begin{aligned} u_{tt}(r,t) &= u_{rr}(r,t) + \frac{1}{r}u_r(r,t), \\ u(2,t) &= 0, \\ u(r,0) &= 0, \\ u_t(r,0) &= 1. \end{aligned}$$

Xc4.2-12a. **(Series Identity for $J_0(x)$)**

Using Bessel's equation of order zero,

$$x^2y'' + xy' + x^2y = 0,$$

derive from the Frobenius method the series formula

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n!)^2} x^{2n}.$$

Observe that this is a special solution obtained from the Frobenius series $y = \sum_{n=0}^{\infty} a_n x^n$ by taking $a_0 = 1$.

Xc4.2-12b. **(Bessel Function Identities)**

Establish the identities

$$\begin{aligned} \int J_1(x)dx &= -J_0(x) + c, \\ \int xJ_0(x)dx &= xJ_1(x) + c. \end{aligned}$$

Week 12: 4.3-4.4 – Non-Symmetric Drumhead and 2D Wave Equation

Prob4.3-3. **(Non-Symmetric Drumhead)**

Solve the drumhead problem for $u(r,\theta,t)$. Plot the drumhead for $t = 0, 1, 3$ using a truncated series of four nonzero numerically approximated coefficients. Assume in the problem statement $0 < r < 2$, $0 < \theta < 2\pi$, $t > 0$.

$$\begin{aligned} u_{tt}(r,\theta,t) &= u_{rr}(r,\theta,t) + \frac{1}{r}u_r(r,\theta,t) + \frac{1}{r^2}u_{\theta\theta}(r,\theta,t), \\ u(2,\theta,t) &= 0, \\ u(r,0,t) &= u(r,2\pi,t), \\ u_\theta(r,0,t) &= u_\theta(r,2\pi,t), \\ u(r,\theta,0) &= (4-r^2)r\sin\theta, \\ u_t(r,\theta,0) &= 1. \end{aligned}$$

Xc4.3-13. **(Two-Dimensional Heat Conduction)**
Solve the heat conduction problem in a circular plate for $u(r,\theta,t)$. Assume in the problem statement $0 < r < 1$, $0 < \theta < 2\pi$, $t > 0$.

$$\begin{aligned} u_t(r,\theta,t) &= u_{rr}(r,\theta,t) + \frac{1}{r}u_r(r,\theta,t) + \frac{1}{r^2}u_{\theta\theta}(r,\theta,t), \\ u(1,\theta,t) &= \sin 3\theta, \\ u(r,\theta,0) &= 0 \\ u_t(r,\theta,0) &= 1. \end{aligned}$$

Prob4.4-5. **(Dirichlet Problem on a Disk)**
Solve the unit disk steady-state heat problem on $0 < r < 1$, $0 \le \theta < 2\pi$ for $u(r,\theta)$:

$$\begin{aligned} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) &= 0, \\ u(1,\theta) &= \begin{cases} 100 & 0 \le \theta \le 0.25\pi, \\ 0 & 0.25\pi < \theta < 2\pi. \end{cases} \end{aligned}$$

Week 13: 4.4 – Steady-State Temperature in a Disk

Prob4.4-15a. **(Exterior Dirichlet Problem on a Disk)**
Solve the unit disk exterior steady-state heat problem on $r > 1$, $0 \le \theta < 2\pi$ for $u(r,\theta)$:

$$\begin{aligned} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) &= 0, \\ u(1,\theta) &= \begin{cases} 100 & 0 \le \theta \le 0.25\pi, \\ 0 & 0.25\pi < \theta < 2\pi. \end{cases} \end{aligned}$$

Xc4.4-11. **(Dirichlet Series Formula)**
Establish the identity

$$\sum_{n=1}^{\infty} \frac{u^n}{n}\sin(n\theta) = \arctan\left(\frac{u\sin\theta}{1 - u\cos\theta}\right).$$

Hint: Let $r = u$ to derive the identity. Take the imaginary part of the Taylor expansion $-\ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ to extract the identity. Use the fact that $w = re^{i\theta} = x + iy$ implies $\mathbf{Real}(\ln w) = \ln r$ and $\mathbf{Imag}(\ln w) = \theta = \arctan(y/x)$.

Prob4.4-15b. **(Cartesian Coordinates)**
Use the *Dirichlet series formula* with $u = 1/r$ to convert to rectangular xy-coordinates the exterior Dirichlet problem solution in polar form

$$u(r,\theta) = \frac{25}{2} + \frac{100}{\pi}\sum_{n=1}^{\infty} \frac{r^{-n}(1-(-1)^n)}{n}\sin(n\theta).$$

Reference: Asmar *PDE and BVP*, section 4.4, where you will find several useful trigonometric identities. Use $(-1)^n = \cos(n\pi)$, $x = r\cos\theta$, $y = r\sin\theta$ for the conversion.

Answer: $u(r,\theta) = \frac{25}{2} + \frac{100}{\pi}(\arctan(v_1) - \arctan(v_2))$ where $v_1 = \dfrac{\sin\theta}{r - \cos\theta}$ and $v_2 = \dfrac{\sin\theta}{r + \cos\theta}$. In terms of x and y, $v_1 = \dfrac{y}{x^2 + y^2 - x}$, $v_2 = \dfrac{y}{x^2 + y^2 + x}$.

Prob4.4-15c. **(Isotherms)**
Consider the exterior Dirichlet problem on the unit disk, $r > 1$, $0 \le \theta < 2\pi$,

$$\begin{aligned} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) &= 0, \\ u(1,\theta) &= \begin{cases} 100 & 0 \le \theta \le 0.25\pi, \\ 0 & 0.25\pi < \theta < 2\pi. \end{cases} \end{aligned}$$

The isotherms are the xy-plane curves of constant temperature described by $u(x,y) = T$, where $0 \le T \le 100$. They are circles. Find their equations and plot a representative set of isotherms.