Name ___________________________

Math 3150 Problems
Weeks 2-3
June 2009

Due date: See the internet due date. Problems are collected twice a week. Records are locked when the stack is returned. Records are only corrected, never appended.

Submitted work. Please submit one stapled package per problem set. Label each problem with its corresponding problem number, e.g., $\boxed{\text{Prob3.1-4}}$ or $\boxed{\text{Xc1.2-4}}$. Kindly label extra credit problems with label $\boxed{\textbf{Extra Credit}}$. You may attach this printed sheet to simplify your work.

Labeling. The label $\boxed{\text{Probx.y-z}}$ means the problem is for chapter $\boxed{\text{x}}$, section $\boxed{\text{y}}$, problem $\boxed{\text{z}}$. When $y = 0$, then the problem does not have a textbook analog, it is a **background problem**. Otherwise, the problem number should match a corresponding problem in the textbook. The same labeling applies to extra credit problems, e.g., $\boxed{\text{Xc1.0-4}}$, $\boxed{\text{Xc1.1-2}}$.

Week 2: 2.1, 2.2, 2.3 – Periodic Functions and Fourier Series

Prob2.0-1. (Trigonometric Identities and Integrals)

(a) Use the trigonometric identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$ to derive the trigonometric identity

$$\cos mx \cos nx = \frac{1}{2}\left(\cos((m+n)x) + \cos((m-n)x)\right).$$

(b) Show the details for integrating $\cos mx \cos nx$ for nonnegative integers $m \neq n$ over $-\pi \leq x \leq \pi$.

(c) Derive the trigonometric identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ from the trigonometric identities $\cos(a+b) = \cos a - \cos b$ and $\cos^2\theta + \sin^2\theta = 1$.

(d) Integrate $\cos^2 nx$ for integers $n = 0, 1, 2, 3, \ldots$ over $-\pi \leq x \leq \pi$. Explain geometrically why there are two different answers.

Prob2.0-2. (Orthogonality)

Two vectors $\vec{A}$, $\vec{B}$ are said to be **orthogonal** if their dot product is zero. For vectors of dimension n, this means $a_1b_1 + a_2b_2 + \cdots + a_nb_n = 0$.

(a) The equation $\int_0^1 f(x)g(x)dx = 0$ can be viewed as *the Riemann sum is approximately zero*. Argue that this means $\vec{A}\cdot\vec{B} = 0$ to so many decimal places, where $\vec{A}$ and $\vec{B}$ are **vector samples** of f and g represented in the Riemann sum $h\sum_{j=1}^n f(jh)g(jh)$, $h = \frac{1}{n}$.

(b) Prove the orthogonality relation below from the standard one for the trigonometric system on $-\pi \leq x \leq \pi$, by a change of variables. The method leads to six orthogonality relations for the trigonometric system $\{\cos(m\pi x/T)\}_{m=0}^\infty$, $\{\sin(n\pi x/T)\}_{n=1}^\infty$ on $-T \leq u \leq T$.

$$\int_{-T}^{T} \cos(m\pi u/T)\cos(n\pi u/T)du = \begin{cases} 0 & m \neq n, \\ T & m = n. \end{cases}$$

Prob2.1-1. (Periodic Functions)

(a) Find the period, amplitude and frequency of $\sin^2(4x)$.

(b) Let f be periodic of period 1 and on $0 \leq x \leq 1$ $f(x) = f_0(x)$, where $f_0(x) = 1$ on $0 \leq x < 1/2$, $f_0(x) = 0$ on $1/2 \leq x < 1$, $f_0(1) = 0$. Graph f on $-2 \leq x \leq 3$.

Prob2.1-8. (Sums of Periodic Functions)

(a) Find the period of $\cos x + \cos 3x$.

(b) Find the period of $e^{2\cos 2x}$.

Xc2.1-8. **(Periodic Functions)**

Explain why $\cos x + \cos 3\pi x$ is not periodic, by analyzing the number of solutions of the equation $\cos x + \cos 2\pi x = 2$. A precise analysis is expected, beginning with a graphic and ending with a mathematical argument [use $-2 \le f(x) \le 2$].

Xc2.1-18. **(Change of Variables)**

(a) Prove that $f(x)$ continuous and T-periodic implies $\displaystyle\int_0^T f(x)dx = \int_{nT}^{nT+T} f(u)du$.

(b) Assume $f(x)$ is 2π-periodic and continuous. Prove that $F(x) = \int_0^x f(u)du$ is 2π-periodic if and only if $\int_0^{2\pi} f(x)dx = 0$.

Xc2.1-20. **(Floor Function)**

The greatest integer function or **staircase function** is represented using a library function **floor**(x), available in most computer mathematical workbenches, including `MATLAB` and `MAPLE`. Don't confuse **floor** with **trunc** – they are different functions!

(a) Plot **floor**(x) and $x -$ **floor**(x) in `MATLAB` or `MAPLE` on $-3 \le x \le 5$. Programs `Excel` and `OpenOffice` can also be used to make the plot. The **floor** function in `Excel` is $x \to \mathrm{FLOOR}(x, 1)$.

(b) Argue from the graphic that $x -$ **floor**(x) is periodic of period 1.

(c) Define $g(x, T) = x - T\,\mathbf{floor}((x + T/2)/T)$. Show mathematically that $g(x) = x$ on $|x| < T/2$. That the **triangular wave** g is T-periodic can be seen from its graphic.

(d) Plot the 2-periodic **triangular wave** defined by $f(x) = |x - 2\,\mathbf{floor}((x + 1)/2)|$.

Problem Notes: The function f in (d) equals $|g(x, T)|$, where $T = 2$ and g is defined in (c) above. The function $a|g(x, T)|$ is called a **triangular wave** of height a and period T. It is the composition of $u \to a|u|$ and $x \to g(x, T)$.

Prob2.2-5. **(Fourier Series Partial Sum Plots)**

(a) Plot on $-2\pi \le x \le 3\pi$ the partial sums $s_2(x), s_6(x), s_{12}(x), s_{20}(x)$ of the Fourier series for the sawtooth wave f constructed from $f_1(x) = |x|$ on $|x| \le \pi$:

$$s_N(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{N} \frac{1}{(2m+1)^2} \cos(2mx + x).$$

The four snapshots show the convergence of the partial sums to the limiting Fourier series.

(b) Explain what happens in the graphic of (a) at points of discontinuity of f'.

Xc2.2-5a. **(Fourier Series Partial Sum Plots)**

(a) Plot on $-2\pi \le x \le 3\pi$ the partial sums $s_2(x), s_6(x), s_{12}(x), s_{20}(x)$ of the Fourier series for the sawtooth wave f constructed from $f_1(x) = (\pi - x)/2$ on $0 < x \le 2\pi$:

$$s_N(x) = \sum_{n=1}^{N} \frac{1}{n} \sin(nx).$$

The four snapshots show the convergence of the partial sums to the limiting Fourier series.

(b) Explain what happens in the graphic of (a) at points of discontinuity of f.

(c) Illustrate Gibb's phenomenon. In particular, graph $|f(x) - s_N(x)|$ on $-2\pi \le x \le 3\pi$, for $N = 6, 12, 20$. Then estimate the jump at points of discontinuity of f'.

Answer: About 1.25.

Prob2.2-8. **(Fourier Series Computation and Graphics)**

Let f be the 2π-periodic extension of the rectified cosine wave $f_0(x) = |\cos x|$ on $|x| \le \pi$.

(a) Draw a graphic of $f(x)$ on $-4\pi \le x \le 5\pi$.

(b) Show the derivation details for the Fourier series $\displaystyle\frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2 - 1} \cos 2mx$.

(c) Plot the Fourier series on $-2\pi \le x \le 2\pi$. Explain why it differs from the plot of $f(x)$ on the same interval.

Xc2.2-15. **(Fourier Series Computation)**

Show the derivation details for the Fourier coefficients of $f(x)$ constructed from $f_2(x) = e^{-|x|}$ on $|x| \le \pi$. The Fourier series is

$$\frac{e^\pi - 1}{\pi e^\pi} + \frac{2}{\pi e^\pi} \sum_{m=1}^{\infty} \frac{e^\pi + (-1)^{m+1}}{m^2+1} \cos mx.$$

Prob2.0-3. **(Even and Odd Functions)**

(a) Define *even function* and *odd function*. Such functions don't have to be continuous, but they must be defined for all x.

(b) Show the mathematical details in the derivation of the result *(Even)(Odd)=Odd*.

(c) Prove by a u-substitution that $\int_{-p}^{p} f(x)dx = 2\int_0^p f(x)dx$ for an even continuous function f and $\int_{-p}^{p} g(x)dx = 0$ for an odd continuous function $g(x)$.

Prob2.3-7. **(Fourier Series Arbitrary Period)**

(a) Define f to be the periodic extension of period 4 of the base function $f_0(x) = 1 - x$ on $0 \le x \le 2$, $f_0(x) = -1 - x$ on $-2 \le x \le 0$. Plot $f(x)$ on $-8 \le x \le 6$.

(b) Show the derivation details for the Fourier series of $f(x)$:

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{2n}.$$

Xc2.3-32. **(Failure of Term-by-Term Differentiation)**

Show that the Fourier series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ of the sawtooth wave f cannot de differentiated term-by-term to obtain the Fourier series of f'.

Xc2.3-34. **(Term-by-Term Integration)**

Integrate the Fourier series of the triangular wave f constructed from $f_0(x) = x$ on $|x| \le 1$, in order to find the Fourier series of the parabolic wave g constructed from $g_0(x) = x^2$ on $|x| \le 1$.

Week 3: 2.4, 2.6 – Fourier Series Methods

Prob2.0-4. **(Periodic Extensions)**

Lemma 1. The function $\mathbf{tw}(x) = x - \mathbf{floor}(x + 1/2)$ is a **triangular wave** of period 1 with shape x on $|x| < 1/2$.

Lemma 2. Given $f_0(x)$ defined on $|x| \le T/2$, then $f(x) = f_0(T\,\mathbf{tw}(x/T))$ is the T-periodic extension of $f_0(x)$ from $|x| \le T/2$ to $-\infty < x < \infty$.

Assume Lemmas 1 and 2 for this problem.

(a) Plot $f_1(x) = 3\,\mathbf{tw}(x/3)$ on $-6 \le x \le 6$. Document its period on the graphic.

(b) Define $f_2(x) = |\cos(0.5\pi\,\mathbf{tw}(2x/\pi))|$. Make a plot on $-2\pi \le x \le 3\pi$. Document its period on the graphic.

Prob2.0-5. **(Even and Odd Periodic Extensions)**

Definition. Define $\mathbf{signum}(x) = \begin{cases} \frac{x}{|x|} & x \ne 0, \\ 0 & x = 0. \end{cases}$.

There is no agreement in literature how to define $\mathbf{signum}(0)$. Here, $\mathbf{signum}(x)$ takes on only the values 1, -1 and 0.

(a) Let $p = 2$ and define $g_1(x) = x^2$ on $0 \le x \le p$. Let $g_2(x) = \mathbf{signum}(x)g_1(|x|)$ be the odd extension of g_1 to $|x| \le p$. Let $T = 2p$. Define $f_3(x) = g_2(T\,\mathbf{tw}(x/T))$ to be the odd extension of $g_2(x)$ from $|x| \le p$ to $-\infty < x < \infty$. Plot f_3 on $|x| \le 5$. Then justify why this works in general, for any p and any g_1.

(b) Let $p = 2$ and define $h_1(x) = x^2$ on $0 \le x \le p$. Let $h_2(x) = h_1(|x|)$ be the even extension of h_1 to $|x| \le p$. Let $T = 2p$. Define $h_3(x) = h_2(T\,\mathbf{tw}(x/T))$ to be the even extension of $h_2(x)$ from $|x| \le p$ to $-\infty < x < \infty$. Plot h_3 on $|x| \le 5$. Then justify why this works in general, for any p and any h_1.

Prob2.0-6. (Dirichlet Kernel Identity)

Establish by trigonometric identity methods the formula [the right side is called **Dirichlet's Kernel**]

$$\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\left(nx + \frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}.$$

Hint: Cross multiply by $2\sin(x/2)$. Expand terms using a trigonometric identity, which produces a telescoping sum.

Prob2.4-7. (Half-Range Expansions)

(a) Find a simple algebraic formula for the even π-periodic extension of $f_0(x) = \cos x$ on $0 \le x \le \pi/2$.

(b) Find the Fourier coefficients for the half-range expansion of $f_0(x) = \cos x$ on $0 \le x \le \pi/2$.

Prob2.4-15. (Half-Range Sine Expansion)

Find the Fourier coefficients for the half-range sine series expansion of e^x on $0 \le x \le 1$.

Prob2.6-6. (Complex Fourier Series)

Find the complex form of the Fourier series for $\sin 3x$ without evaluating any trigonometric integrals.

Hint: Use $\sin u = \frac{1}{2i}\left(e^{iu} - e^{-iu}\right)$.

Prob2.6-11. (Series Identities)

Let $x = 0$ in the complex Fourier series expansion of e^x in order to prove the formula

$$\frac{2\pi}{e^{\pi} - e^{-\pi}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + 1}.$$