Name ___________________________

Math 3150 Problems
Haberman Chapter H4

Due Date: Problems are collected on Wednesday.

Chapter H4: 4.2, 4.3, 4.4 Vibrating String, Boundary Conditions, Fixed Ends BVP

EXERCISES H4.2

Problem H4.2-1. (One-Dimensional String Derivation)

(a) Using the equation

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\,\frac{\partial^2 u}{\partial x^2} + Q(x,t)\rho_0(x),$$

compute the sagged equilibrium position $u_E(x)$ if $Q(x,t) = -g$. Use boundary conditions $u(0) = 0$ and $u(L) = 0$ with the equilibrium equation $0 = T_0\frac{d^2u}{dx^2} - g\rho_0$ (formally, $u_{tt} \equiv 0$).
(b) Show that $v(x,t) = u(x,t) - u_E(x)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T_0}{\rho_0(x)}.$$

Use superposition and the equation $0 = u_{xx} - gc^2$, valid for $u = U_E$ according to part (a).

Problem H4.2-2. (Wave Speed c)

Show that $c^2 = \dfrac{T_0}{\rho_0}$ has the dimensions of velocity squared.

Problem XC-H4.2-3. (String Derivation)

Consider a particle whose x-coordinate (in horizontal equilibrium) is designated by α. If its vertical and horizontal displacements are u and v, respectively, determine its position x and y. Then show that

$$\frac{dy}{dx} = \frac{\partial u/\partial\alpha}{1 + \partial v/\partial\alpha}.$$

Problem XC-H4.2-4. (String Derivation)

Derive equations for horizontal and vertical displacements without ignoring v. Assume that the string is perfectly flexible and that the tension is determined by an experimental law.

Problem XC-H4.2-5. (String Derivation, Constant Wave Speed)

Derive the partial differential equation for a vibrating string in the simplest possible manner. You may assume the string has constant mass density ρ_0, you may assume the tension T_0 is constant, and you may assume small displacements (with small slopes).

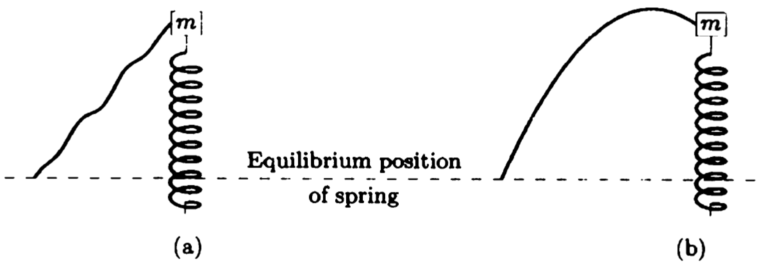
EXERCISES H4.3

Problem XC-H4.3-1. **(Boundary Conditions)**

If $m = 0$ in the model

$$m\frac{d^2u}{dt^2}(0,t) = -k(u(0,t) - y_s(t) - L) + T_0\frac{\partial u}{\partial x}(0,t) + g(t)$$

then which of the diagrams for the right end of the string shown in the figure can possibly be correct? *Briefly* explain. Assume that the mass can move only vertically.



EXERCISES H4.4

Problem H4.4-1. **(One-Dimensional String, Constant Wave Speed c)**

Consider vibrating strings of uniform density ρ_0 and tension T_0. Submit only (a) and (b).

*(a) What are the natural frequencies of a vibrating string of length L fixed at both ends?

*(b) What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u/\partial x(H,t) = 0$]? Sketch a few modes of vibration as in Fig. 1, H4.4.

(c) Show that the modes of vibration for the odd harmonics (i.e., $n = 1, 3, 5, \ldots$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

Problem H4.4-2. **(Time-Dependent String Equation)**

In Sec. H4.2 it was shown that the displacement u of a nonuniform string satisfies

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + Q(x,t)\rho_0(x),$$

where Q represents the vertical component of the body force per unit length. If $Q = 0$, the partial differential equation is homogeneous. A slightly different homogeneous equation occurs if $Q = \alpha u$.

Submit only part (c).

(a) Show that if $\alpha < 0$, the body force is restoring (toward $u = 0$). Show that if $\alpha > 0$, the body force tends to push the string further away from its unperturbed position $u = 0$.

(b) Separate variables if $\rho_0(x)$ and $\alpha(x)$ are assumed to depend on x, but T_0 is constant for physical reasons. Analyze the time-dependent ordinary differential equation.

*(c) Specialize part (b) to the constant coefficient case. Solve the initial value problem if $\alpha < 0$:

$u(0,t) = 0,\ u(L,t) = 0,$

$u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = f(x).$

What are the frequencies of vibration?

Problem XC-H4.4-3. (Damped Vibrations of a One-Dimensional String)

Consider a slightly damped vibrating string that satisfies

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\,\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

*(b) Determine the solution (by separation of variables) that satisfies the boundary conditions
$u(0,t) = 0$ and $u(L,t) = 0$
and the initial conditions
$u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$-
You can assume that this frictional coefficient Q is relatively small ($\beta^2 < 4\pi^2\rho_0 T_0/L^2$).

Problem XC-H4.4-4. (Eigenfunction Expansion Method)

Redo Exercise 3(b), H4.4, by the eigenfunction expansion method.

Problem XC-H4.4-5. (Damped Vibrations)

Redo Exercise 3(b), H4.4, if $4\pi^2\rho_0 T_0/L^2 < \beta^2 < 16\pi^2\rho_0 T_0/L^2$.

Problem XC-H4.4-6. (d'Alembert Solution)

For the classical string vibration problem with clamped ends, use the series solution for $u(x,t)$ to show that

$$u(x,t) = R(x-ct) + S(x+ct),$$

where R and S are some functions.

Problem H4.4-7. (d'Alembert Solution)

If a vibrating string satisfying the one-dimensional string equation with fixed ends is initially at rest, $g(x) = 0$, with shape $f(x)$ given, then show that

$$u(x,t) = \frac{1}{2}[F(x-ct) + F(x+ct)],$$

where $F(x)$ is the odd periodic extension of $f(x)$.

Hints.
1. For all x, $F(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L)$.
2. $\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$.

Comment: This result shows that the practical difficulty of summing an infinite number of terms of a Fourier series may be avoided for the one-dimensional wave equation.

Problem XC-H4.4-8. (d'Alembert Solution)

If a vibrating string satisfying the one-dimensional string equation with fixed ends is initially unperturbed, $f(x) = 0$, with the initial velocity $g(x)$ given, then show that

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(u)du,$$

where $G(x)$ is the odd periodic extension of $g(x)$.

Hints:
1. For all x, $G(x) = \sum_{n=1}^{\infty} B_b \frac{n\pi c}{L} \sin(n\pi x/L)$.
2. $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$.

See the comment after Exercise 7, H4.4.

Problem XC-H4.4-9. (Energy Conservation)

From $u_{tt} = c^2 u_{xx}$, derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2\ u_x(x,t)u_t(x,t)|_{x=0}^{x=L},$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2}(u_t)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2}(u_x)^2 dx$.

Problem XC-H4.4-10. **(Total Energy)**

What happens to the total energy E of a vibrating string (see Exercise 9, H4.4)
(a) If $u(0,t) = 0$ and $u(L,t) = 0$
(b) If $u_x(0,t) = 0$ and $u(L,t) = 0$
(c) If $u(0,t) = 0$ and $u_x(L,t) = -\gamma u(L,t)$ with $\gamma > 0$
(d) If $\gamma < 0$ in part (c)

Problem H4.4-11. **(Potential and Kinetic Energies)**

Show that the potential and kinetic energies (defined in Exercise 9, H4.4) are equal for a traveling wave, $u = R(x - ct)$.

Problem XC-H4.4-12. **(Uniqueness)**

Using

$$\frac{dE}{dt} = c^2\, u_x(x,t)u_t(x,t)\big|_{x=0}^{x=L},$$

prove that the solution of the one-dimensional string equation with fixed ends is unique.

Remark. The result means that the solution can be computed with numerical software.

Problem XC-H4.4-13. **(Modes and Energies)**

(a) Using

$$\frac{dE}{dt} = c^2\, u_x(x,t)u_t(x,t)\big|_{x=0}^{x=L},$$

calculate the energy of one normal mode.

(b) Show that the total energy, when $u(x,t)$ is a superposition of product solutions representing the solution, is the sum of the energies contained in each mode.